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Controllability of certain real symmetric matrices with application to controllability of graphs

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Abstract

If M is an $n \times n$ real symmetric matrix and $\mathbf{b}$ is a real vector of length n, then the pair $(M, \mathbf{b})$ is said to be controllable if all the eigenvalues of M are simple and M has no eigenvector orthogonal to $\mathbf{b}$. Simultaneously, we say that M is controllable for $\mathbf{b}$. There is an extensive literature concerning controllability of specified matrices, and in the recent past the matrices associated with graphs have received a great deal of attention. In this paper, we restate some known results and establish new ones related to the controllability of similar, commuting or Gram matrices. Then we apply the obtained results to get an analysis of controllability of some standard matrices associated with (particular) graphs.

Keywords: eigenvalues and eigenvectors; controllability; similar matrices; commuting matrices; Gram matrix.

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1. Introduction

The following differential equation is a standard model for the single-input linear control systems:

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x} + \mathbf{b}u. \tag{1}$$

The scalar $u = u(t)$ is called the *control input*, while $M \in \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. The system (1) is said to be *controllable* if, for a vector $\mathbf{x}^*$ and time t^*, there always exists a function $u(t)$, $0 < t < t^*$, such that the solution of (1) gives $\mathbf{x}(t^*) = \mathbf{x}^*$ irrespective of $\mathbf{x}(0)$. Controllability plays a significant role in control problems such as stabilization of unstable systems or optimal control. It also has applications in reachability theory or viability theory. For more details, we refer to [7].

In general, we do not assume any special structure or property of the matrix M or the vector $\mathbf{b}$. However, the case where M is a symmetric matrix has received a significant attention in the recent past (see [1,4,9,11] and references cited therein). In this case, the system (1) is controllable if and only if the matrix

$$[\mathbf{b} \,|\, M\mathbf{b} \,|\, \cdots \,|\, M^{n-1}\mathbf{b}] \tag{2}$$

has full rank. It follows that the system is controllable if M has no repeated eigenvalues and the corresponding eigenvectors are non-orthogonal to $\mathbf{b}$. For details, see any of the mentioned references.

We say that the pair $(M, \mathbf{b})$ is *controllable* and also that M is *controllable for* $\mathbf{b}$ if the corresponding system is controllable. If M is some standard matrix associated with a graph, then we simplify language by saying that the graph under consideration is controllable (for the corresponding matrix and the vector $\mathbf{b}$).

In this study we restrict ourselves to symmetric matrices (although some results remain valid for some wider classes, say for diagonalizable matrices). In particular, we consider the controllability of similar, commuting or Gram matrices. As an application, we consider the controllability of some particular graphs.

2. Similar matrices

Throughout this and the next section, we assume that the matrices M and N are real and symmetric. In this section, let M and N be $n \times n$ similar matrices, that is let $M = P^{-1}NP$ for some invertible matrix P. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\intercal$ is an eigenvector associated with the eigenvalue λ of M, then $P\mathbf{x}$ is associated with the same eigenvalue of N. (Note that $P\mathbf{x} \neq \mathbf{0}$, since P is invertible.)

For an $n \times 1$ vector $\mathbf{b} = (b_1, b_2, \ldots, b_n)^\intercal$, $(M, \mathbf{b})$ and $(N, \mathbf{b})$ are controllable if and only if M has no repeated eigenvalues and $\mathbf{x} \cdot \mathbf{b} \neq 0$ and $P\mathbf{x} \cdot \mathbf{b} \neq 0$, for all eigenvectors $\mathbf{x}$ of M.

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In particular, if $M = D^{-1}ND$, for a diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, then $(M, \mathbf{b})$ and $(N, \mathbf{b})$ are controllable if and only if M has no repeated eigenvalues and $\sum_{i=1}^{n} x_i b_i \neq 0, \sum_{i=1}^{n} x_i d_i b_i \neq 0$, for all eigenvectors $\mathbf{x}$ of M.

Here is another case.

Proposition 2.1. *If $M = P^{-1}NP$, then $(M, \mathbf{b})$ is controllable if and only if $(N, P\mathbf{b})$ is controllable.*

Proof. If $(M, \mathbf{b})$ is controllable, then $[\mathbf{b} \,|\, M\mathbf{b} \,|\, \cdots \,|\, M^{n-1}\mathbf{b}]$ has full rank, but then

$$[P\mathbf{b} \,|\, NP\mathbf{b} \,|\, \cdots \,|\, N^{n-1}P\mathbf{b}] = P[\mathbf{b} \,|\, M\mathbf{b} \,|\, \cdots \,|\, M^{n-1}\mathbf{b}]$$

also has full rank (since P is invertible), and the result follows. □

The last proposition is known from literature, see [7]. Here is a simple corollary.

Corollary 2.1. *If $M = D^{-1}ND$ for a diagonal matrix D with ± 1s on the main diagonal, then $(M, \mathbf{b})$ is controllable if and only if $(N, D\mathbf{b})$ is controllable.*

The previous corollary is significant in study of controllability in the category of signed graphs (i.e., graphs whose edges are accompanied either by 1 or -1), since it gives a method to consider controllability within a switching equivalence class. Namely, we say that signed graphs $\dot{G}$ and $\dot{H}$ are *switching equivalent* if there is a vertex labelling such that their adjacency matrices satisfy $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$, for some D defined in the corollary. Consequently, controllability of one of them determines controllability of the other.

We proceed with the controllability of M in general case.

Proposition 2.2. *The pair $(M, \mathbf{b})$ is controllable if and only if M has no repeated eigenvalues and $\mathbf{b} = X\mathbf{s}$, where the columns of X are linearly independent normalized eigenvectors of M and $\mathbf{s}$ has no zero coordinate.*

Proof. The matrix M is diagonalizable and we have $D = X^{-1}MX$, where D is the diagonal matrix with the eigenvalues of M on the main diagonal and X is defined in the theorem.

Considering the controllability of D, since its eigenvectors are the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of the canonical basis, we conclude that $(D, \mathbf{s})$ is controllable if and only if $\mathbf{s}$ is non-orthogonal to any $\mathbf{e}_i$, equivalently $\mathbf{s}$ is not spanned by any proper subset of $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, equivalently $\mathbf{s}$ has no zero coordinate. Now, the result follows by Proposition 2.1. □

In other words, if M has no repeated eigenvalues, then all controllable pairs $(M, \mathbf{b})$ are easily determined by a full set of linearly independent eigenvectors of M. Here is a consequence.

Corollary 2.2. *With notation of Proposition 2.2, let $\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^{\intercal}$, $1 \leq i \leq n$, be the columns of X, i.e., the normalized eigenvectors of M. If M has no repeated eigenvalues and $\mathbf{b} = (\sum_{j=1}^{n} x_{j1}, \sum_{j=1}^{n} x_{j2}, \ldots, \sum_{j=1}^{n} x_{jn})^{\intercal}$, then $(M, \mathbf{b})$ is controllable.*

Proof. The result follows by taking $\mathbf{s}$ to be the all-1 vector of $\mathbb{R}^n$. □

We remark that if M is the adjacency matrix of a graph G, then the matrix (2) is known as the *walk matrix* of G. The walk matrix with $\mathbf{b}$ as described in the previous corollary is called a *closed pseudo walk matrix*. Due to Farrugia [6], the rank of this matrix is equal to the number of distinct eigenvalues of G.

Using Proposition 2.2, we compute all vectors $\mathbf{b}$ which preserve the controllability of paths P_n and the Laplacian controllability of threshold graphs T_n. We denote by A_G and L_G the adjacency matrix and the Laplacian matrix of a graph G, respectively.

Proposition 2.3. *For $\mathbf{b} = (b_1, b_2, \ldots, b_n)^{\intercal}$, $(A_{P_n}, \mathbf{b})$ is controllable if and only if*

$$b_i = \sum_{j=1}^{n} \frac{s_j}{n_j} \sin \frac{ij\pi}{n+1},$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_n)^{\intercal}$ is an arbitrary vector with no zero coordinates and

$$n_i = \frac{1}{2}\sqrt{2n + 1 - \frac{\sin\frac{(2n+1)i\pi}{n+1}}{\sin\frac{i\pi}{n+1}}}.$$

Proof. It is well known that the eigenvalues of P_n are $2\cos\frac{i\pi}{n+1}$, for $1 \leq i \leq n$. Thus, all of them are simple. It follows by direct computation that a full system of linearly independent eiegnvectors of P_n is given by

$$\mathbf{x}_i = \Big(\sin\frac{i\pi}{n+1}, \sin\frac{2i\pi}{n+1}, \dots, \sin\frac{ni\pi}{n+1}\Big)^\intercal, \text{ for } 1 \leq i \leq n.$$

This result can also be derived from [2, Theorem 3.7]. To obtain the normalized eigenvectors, we compute

$$\begin{aligned}\sum_{k=1}^{n}\sin^2\frac{ki\pi}{n+1} &= \sum_{k=1}^{n}\frac{1-\cos\frac{2ki\pi}{n+1}}{2} = \frac{1}{2}\Big(n - \sum_{k=1}^{n}\cos\frac{2ki\pi}{n+1}\Big)\\ &= \frac{1}{2}\Bigg(n + \frac{1}{2} - \frac{\sin((n+\frac{1}{2})\frac{2i\pi}{n+1})}{2\sin\frac{i\pi}{n+1}}\Bigg) = \frac{1}{4}\Bigg(2n+1-\frac{\sin\frac{(2n+1)i\pi}{n+1}}{\sin\frac{i\pi}{n+1}}\Bigg),\end{aligned}$$

where the sum of cosines is computed by the Lagrange's trigonometrical identity. Therefore, the norm of $\mathbf{x}_i$ is equal to n_i. The result follows by Proposition 2.2, along with an observation that the eigenvectors of P_n can be arranged into the matrix X (of the same proposition) in an arbitrary way (by using an appropriate rearrange of the coordinates of $\mathbf{s}$). □

Recall that a *threshold graph* is defined as a graph which does not contain any of $2K_2, P_4$ or C_4 (i.e., the union of two edges, the path with 4 vertices and the cycle with 4 vertices) as an induced subgraph. Every threshold graph T_n can be generated by so-called *binary generating sequence* $(a_1, a_2, \dots, a_n)$ in the following way:

1. For $i = 1$, $T_1 = T(a_1) = K_1$;
2. For $i \geq 2$, with $T_{i-1} = T(a_1, a_2, \dots, a_{i-1})$, T_i is formed by adding an isolated vertex to T_{i-1} if $a_i = 0$ or by adding a vertex adjacent to all the vertices of T_{i-1} if $a_i = 1$.

Without loss of generality, we may assume that $a_1 = 0$. Here is the result.

Proposition 2.4. *For* $\mathbf{b} = (b_1, b_2, \dots, b_n)^\intercal$, $(L_{T_n}, \mathbf{b})$ *is controllable if and only if* T_n *is generated by either* $(0, 1, 0, 1, \dots, 0, 1)$ *or* $(0, 0, 1, 0, 1, \dots, 0, 1)$ *and*

$$b_i = \sqrt{\frac{i-1}{i}}s_{i-1} - \sum_{j=i}^{n-1}\frac{1}{\sqrt{j(j+1)}}s_j + \frac{1}{\sqrt{n}}s_n,$$

where s_0 *is an arbitrary real number and* $s_i \neq 0$*, for* $1 \leq i \leq n$*.*

Proof. First, T_n has no repeated eigenvalues if and only if it is generated by one of the given sequences, as follows by the result of Merris [8, Theorem 2.1]. This fact is explicitly proved in [1, 10]. Next, the eigenvectors of such a T_n are given by

$$\mathbf{x}_i = (\underbrace{-1, -1, \dots, -1}_{i}, i, 0, 0, \dots, 0)^\intercal, \text{ for } 1 \leq i \leq n-1,$$

and $\mathbf{x}_n = \mathbf{j}$, the all-1 vector. This also follows by the same result of [8], since T_n is constructed by a consecutive application of the join operation – see the two mentioned references, as well. By normalizing the eigenvectors and using Proposition 2.2, we get the desired result. □

3. Commuting matrices

We state the following.

Proposition 3.1. *If an* $n \times n$ *matrix* M *has no repeated eigenvalues and*

$$p(M) = a_0 I + a_1 M + \cdots + a_{n-1}M^{n-1}, \tag{3}$$

then $(p(M), \mathbf{b})$ *is controllable if and only if* $(M, \mathbf{b})$ *is controllable and the Vandermonde matrix*

$$\begin{bmatrix} 1 & p(\lambda_1) & p(\lambda_1)^2 & \cdots & p(\lambda_1)^{n-1}\\ 1 & p(\lambda_2) & p(\lambda_2)^2 & \cdots & p(\lambda_2)^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & p(\lambda_n) & p(\lambda_n)^2 & \cdots & p(\lambda_n)^{n-1}\end{bmatrix},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ *are the eigenvalues of* M*, is invertible.*

Proof. Since M has no repeated eigenvalues, by the Frobenius theorem, a matrix N commutes with M if and only if $N \in \mathrm{Span}\{I, M, \dots, M^{n-1}\}$. In other words, $Mp(M) = p(M)M$, which means that M and $p(M)$ share the same eigenvectors. Therefore, the controllability of $(p(M), \mathbf{b})$ implies the controllability of $(M, \mathbf{b})$. To conclude the proof we need to show that $p(M)$ has no repeated eigenvalues if and only if the corresponding Vandermonde matrix is invertible. Since all the possible matrices on the right-hand side of (3) commute, they can be simultaneously diagonalized, which means that the eigenvalues of $p(M)$ are $p(\lambda_1), p(\lambda_2), \dots, p(\lambda_n)$. The statement follows since the determinant of the Vandermonde matrix is given by $\prod_{i<j}(p(\lambda_i) - p(\lambda_j))$. □

The dimension of the vector space of matrices which commute with an $n \times n$ matrix M without repeated eigenvalues is n, since the minimal polynomial of M is precisely its characteristic polynomial. Thus, if q is a real polynomial of an arbitrary degree, we have that $q(M)$, which obviously commutes with M, belongs to $\mathrm{Span}\{I, M, \dots, M^{n-1}\}$.

Haemers and Omidi [5] defined a *universal adjacency matrix* U of a graph as a linear combination of the adjacency matrix, the diagonal matrix of vertex degrees, the identity matrix and the all-1 matrix, along with a non-zero coefficient for the adjacency matrix. Some standard graph matrices (as the adjacency matrix, the Laplacian matrix or the signless Laplacian matrix) are obtained by taking particular coefficients in definition of U.

Proposition 3.2. *Assume that a universal matrix U of a graph has no repeated eigenvalues.*

(i) If N commutes with U, then $(U + N, \mathbf{b})$ is controllable if and only if $(U, \mathbf{b})$ is controllable and $U + N$ has no repeated eigenvalues.

(ii) If N commutes with U, then $(UN, \mathbf{b})$ is controllable if and only if $(U, \mathbf{b})$ is controllable and UN has no repeated eigenvalues.

(iii) For the all-1 matrix J, $J \in \mathrm{Span}\{I, U, \dots, U^{n-1}\}$ if and only if U has constant row sums. In addition, $(U + J, \mathbf{b})$ is controllable if and only if $(U, \mathbf{b})$ is controllable and U has no eigenvalue equal to its row sum plus n.

Proof. (i) follows by Proposition 3.1 (since the assumption that $U + N$ has no repeated eigenvalues is equivalent to that on invertibility of the corresponding Vandermonde matrix). (ii) follows in the same way since UN commutes with U, and thus belongs to the span of $I, U, \dots, U^{n-1}$. Both implications of (iii) are based on the fact that J commutes with U if and only if U has constant row sums, and the latter statement follows since the eigenvalues of $U + J$ are $\lambda_1 + n, \lambda_2, \dots, \lambda_n$, where λ_1 is the eigenvalue of U equal to the row sum (and associated with the all-1 eigenvector). □

One may observe that the previous proposition holds for some other graph matrices, say for the distance matrix.

By (iii), if A $(= A_G)$ is the adjacency matrix of a graph, then $J \in \mathrm{Span}\{I, A, \dots, A^{n-1}\}$ if and only if the corresponding graph is regular. The possibility eliminated by the last assumption (that U has no eigenvalue equal to its row sum plus n) may occur in the case of the Laplacian matrix of a join of two regular graphs, cf. [8].

If A commutes with the adjacency matrix $\overline{A}$ of the complementary graph, then $A(J - I - A) = (J - I - A)A$, which means that $AJ = JA$, and so, by (iii), the corresponding graph is regular. Taking into account that the eigenvalues of $\overline{A}$ are $n - \lambda_1 - 1, -\lambda_2 - 1, \dots, -\lambda_n - 1$ (λ_1 being the vertex degree), we arrive at the following corollary.

Corollary 3.1. *The adjacency matrix A commutes with $\overline{A}$ if and only if the corresponding graph is regular. In this case, if A has no repeated eigenvalues, then $(\overline{A}, \mathbf{b})$ is controllable if and only if $(A, \mathbf{b})$ is controllable and $\lambda_1 - n$ is not an eigenvalue of A.*

The assumption that $\lambda_1 - n$ is not an eigenvalue of A may be replaced by 'G is not the join of two regular graphs'.

4. Gram matrices

Let $S = (\mathbf{s}_1 \,|\, \mathbf{s}_2 \,|\, \cdots \,|\, \mathbf{s}_m)$ be a matrix whose columns are vectors of $\mathbb{R}^m$. Then $S^\intercal S$ is the *Gram matrix* of the inner products $\mathbf{s}_i \cdot \mathbf{s}_j$, for $1 \le i, j \le m$. If $\mathbf{x}$ is an eigenvector associated with a non-zero eigenvalue λ of $S^\intercal S$, then from $S^\intercal S\mathbf{x} = \lambda\mathbf{x}$, we get $SS^\intercal S\mathbf{x} = \lambda S\mathbf{x}$, which means that $S\mathbf{x}$ is associated with the same eigenvalue of $SS^\intercal$.

Consequently, a necessary condition for simultaneous controllability of $(S^\intercal S, \mathbf{b})$ and $(SS^\intercal, \mathbf{b})$ is $\mathbf{x} \cdot \mathbf{b} \neq 0$ and $S\mathbf{x} \cdot \mathbf{b} \neq 0$, for all eigenvectors which are associated with non-zero eigenvalues of $S^\intercal S$. In what follows, we consider a particular situation where S is the vertex-edge incidence matrix of a graph G and $\mathbf{b} = \mathbf{j}$. In this case, $S^\intercal S$ is the adjacency matrix of the line graph $L(G)$, while $SS^\intercal$ is the signless Laplacian matrix Q_G of G.

For G connected, we know from [3] that if $(L(G), \mathbf{j})$ is controllable, then G is a tree or a non-bipartite unicyclic graph. Here we provide the following result.

Proposition 4.1. *With the introduced notation:*

(i) If G is a tree, then $(Q_G, \mathbf{j})$ is controllable if and only if $(L(G), \mathbf{j})$ is controllable and its colour classes of G differ in size;

(ii) If G is a non-bipartite unicyclic graph, then $(Q_G, \mathbf{j})$ is controllable if and only if $(L(G), \mathbf{j})$ is controllable.

Proof. Since every column of S contains exactly two 1s, we have $S\mathbf{x} \cdot \mathbf{j} = 2\mathbf{x} \cdot \mathbf{j}$, for any vector $\mathbf{x}$. In particular, if $\mathbf{x}$ is associated with a non-zero eigenvalue of $L(G)$, then we have $\mathbf{x} \cdot \mathbf{j} \neq 0$ if and only if $S\mathbf{x} \cdot \mathbf{j} \neq 0$. This immediately gives (ii), since in this case zero is not an eigenvalue of Q_G (as the signless Laplacian matrix of a connected graph is singular if and only if the graph is bipartite) and Q_G shares the spectrum with $L(G)$.

For (i), the non-zero eigenvalues of Q_G form the spectrum of $L(G)$, and so it remains to consider the eigenvector, say $\mathbf{y}$, associated with (simple eigenvalue) zero in Q_G. Since $Q_G\mathbf{y} = SS^\intercal\mathbf{y} = \mathbf{0}$, we have $\mathbf{y}^\intercal SS^\intercal\mathbf{y} = \mathbf{0}$, which yields $||S^\intercal\mathbf{y}|| = 0$, i.e., $S^\intercal\mathbf{y} = \mathbf{0}$. It follows that for every edge ij of G the corresponding coordinates of $\mathbf{y}$ satisfy $y_i = -y_j$. Therefore, $\mathbf{y}$ is constant on each colour class and differs in sign on different colour classes. Thus, $\mathbf{y} \cdot \mathbf{j} \neq 0$ if and only if the colour classes differ in size, and the proof is complete. □

In [3] all the controllable pairs $(A_G, \mathbf{j})$, where the least eigenvalue of A_G is not less than -2, are determined, unless G is the line graph of a tree or the line graph of an odd unicyclic graph. By the previous proposition, the controllability in the unsolved cases can be considered by means of the signless Laplacian matrix of G.

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References

[1] C. O. Aguilar, B. Gharesifard, Laplacian controllability classes for threshold graphs, *Linear Algebra Appl.* **471** (2015) 575–586.
[2] R. B. Bapat, *Graphs and Matrices*, Springer, Berlin, 2014.
[3] D. Cvetković, P. Rowlinson, Z. Stanić, M.-G. Yoon, Controllable graphs with least eigenvalue at least -2, *Appl. Anal. Discrete Math.* **5** (2011) 165–175.
[4] C. D. Godsil, Controllable subsets in graphs, *Ann. Comb.* **16** (2012) 733–744.
[5] W. H. Haemers, G. R. Omidi, Universal adjacency matrices with two eigenvalues, *Linear Algebra Appl.* **435** (2011) 2520–2529.
[6] A. Farrugia, On pseudo walk matrices, *Discrete Math. Lett.* **1** (2019) 8–15.
[7] J. Klamka, *Controllability of dynamical systems*, Kluwer Academic Publishers, Dordreht, 1991.
[8] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.* **278** (1998) 221–236.
[9] A. Rahmani, M. Ji, M. Mesbahi, M. Egerstedt, Controllability of multi-agent systems from a graph theoretic perspective, *SIAM J. Control Optim.* **48** (2009) 162–186.
[10] Z. Stanić, Laplacian controllability for graphs with integral Laplacian spectrum, Submitted.
[11] M.-G. Yoon, D. Cvetković, P. Rowlinson, Z. Stanić, Controllability of multy-agent dynamical systems with a broadcasting control signal, *Asian J. Control* **16** (2014) 1066–1072.