SOME NOTES ON SPECTRA OF COGRAPHS

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Abstract

A cograph is a P_4–free graph. We first give a short proof of the fact that 0 (-1) belongs to the spectrum of a connected cograph (with at least two vertices) if and only if it contains duplicate (resp. coduplicate) vertices. As a consequence, we next prove that the polynomial reconstruction of graphs whose vertex–deleted subgraphs have the second largest eigenvalue not exceeding $\frac{\sqrt{5}-1}{2}$ is unique.

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[2] *Keywords:* cograph, characteristic polynomial, eigenvalues, polynomial reconstruction, σ–graph.

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1 Introduction

A *cograph* is usually defined as a P_4–free graph. There are many other definitions of cographs. For example, a cograph is a graph defined by the following rules:

(i) K_1 is a cograph;

(ii) if G and H are cographs then their (disjoint) union $G \oplus H$ is a cograph;

(iii) if G and H are cographs then their join $G \otimes H$ is a cograph.

If G is an arbitrary (simple) graph and u its vertex, then $\Gamma(u)$ and $\Gamma[u]$ denote open and closed neighbourhoods of u, respectively; so $\Gamma(u) = \{v \mid v \sim u\}$ while $\Gamma[u] = \Gamma(u) \cup \{u\}$. Two vertices are *duplicate* (*coduplicate*) if their open (resp. closed) neighbourhoods are the same. So u and v are duplicate (coduplicate) vertices if $\Gamma(u) = \Gamma(v)$ (resp. $\Gamma[u] = \Gamma[v]$).

It was proved by G. Royle (see [10]), that 0 is an eigenvalue of a connected cograph (with at least two vertices) only if it contains duplicate vertices. In addition, it is implicitly mentioned (in the same paper) that -1 is an eigenvalue of a cograph only if it contains coduplicate vertices. In Section 3, we prove both statements in another way.

We now introduce the polynomial reconstruction problem. Let G be a graph on n vertices, and let

$$P_G(x) = \det(xI - A_G) = x^n + a_{n-1}(G)x^{n-1} + \cdots + a_1(G)x + a_0(G)$$

be the characteristic polynomial of its adjacency matrix A_G. Since $P_G(x)$ is invariant with respect to the labelling of the vertices, it is also called the *characteristic polynomial* of G. The collection of eigenvalues of G, i.e.

$$\{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$$

is called the spectrum of G. In sequel, we will usually suppress the graph name from our notation, and in addition will assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $G_i = G - v_i$ $(i = 1, 2, \ldots, n)$, and let

$$\mathcal{P}(G) \ = \ \{P_{G_1}, P_{G_2}, \ldots, P_{G_n}\},$$

be the collection of characteristic polynomials of vertex–deleted subgraphs of G. $\mathcal{P}(G)$ is also called the *polynomial deck* of G; the corresponding collection of subgraphs will be referred to as a *deck*. We consider the following problem.

Problem 1 *Is it true (for $n > 2$) that the characteristic polynomial P_G of G is determined uniquely by its polynomial deck, i.e. by $\mathcal{P}(G)$? In other words, if $\mathcal{P}(G) = \mathcal{P}(H)$, does it mean that $P_G(x) = P_H(x)$ for every x?*

This problem is called *the polynomial reconstruction problem*, and it was posed by D.M. Cvetković (in 1973), and later studied by many authors. In Section 4, we consider the polynomial reconstruction problem for graphs whose second largest eigenvalue is around $\frac{\sqrt{5}-1}{2}$. More precisely, we will assume that all subgraphs from the deck have the second largest eigenvalue bounded from above by σ $(= \frac{\sqrt{5}-1}{2})$, the *golden section*. The structure of graphs G with $\lambda_2(G) \leq \sigma$ has been studied in [7, 8, 14]. Graphs having the property $\lambda_2(G) \leq \sigma$ (σ–property) will be called σ–graphs. For convenience, graphs G for which $\lambda_2(G) < \sigma$ ($\lambda_2(G) = \sigma$) will be called σ^-–graphs (resp. σ^0–graphs), In addition, since every σ^-–graph is a P_4–free graph (by the Interlacing Theorem – see, for example, [3] p. 19), it is as well a cograph.

2 Preliminaries

In view of the second definition of cographs we have that graph G is a cograph if and only if it can be represented by a *cotree* (see [1]). We now define two types of cotrees, denoted by T_G and $\widehat{T}_G$, respectively, representing a cograph G.

The first cotree T_G is a rooted tree (with r as the root) in which any interior vertex w is either of $\oplus$–type (corresponding to the union), or $\otimes$–type (corresponding to the join). The terminal vertices (leaves) are typeless (each of them represents itself in G). Any interior vertex, say w, represents a subgraph of G induced by the terminal successors of w, and is denoted by G_w. All terminal vertices are on the same distance from the root. Also, all interior vertices which are on the same distance from the root are of the same type. Moreover, the direct successors (or children) of any interior vertex w have a type which differs from the type of w (or they are typeless if being the terminal vertices). The direct successors of w (denoted by $w_1, w_2, \ldots, w_q$) represent the subgraphs $G_{w_1}, G_{w_2}, \ldots, G_{w_q}$. If w is of $\oplus$–type then $G_w = \sum_{i=1}^{q} G_{w_i}$, or otherwise, if w is of $\otimes$–type then $G_w = \prod_{i=1}^{q} G_{w_i}$. In particular, $G = G_r$. A *next-to-terminal vertex* (for short, an NTT–vertex) is a vertex of T_G whose all direct successors are terminal vertices.

The second cotree $\widehat{T}_G$ will be called a *minimal cotree*. It is obtained from the previous one by deleting the superfluous vertices, i.e. those interior vertices which have exactly one child. In this situation, its parent (if any) and child are identified. Note also, that in this way, all paths from the root to NTT–vertices are $(\otimes, \oplus)$–alternating, but not necessarily of the same length.

It is worth mentioning that the second representation is unique. As an illustration, we give a simple example (in Fig. 1 we present a cograph G followed by its two representations, T_G and $\widehat{T}_G$).

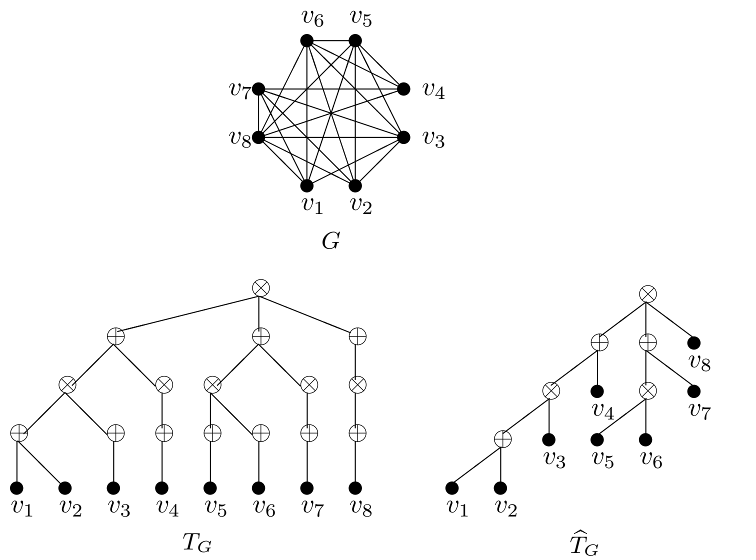

Fig. 1

Remark 2.1 *Note first that minimal cotree $\widehat{T}_G$ enables us to identify easily (in G) the collections of duplicate and coduplicate vertices. Namely, any collection of mutually duplicate (coduplicate) vertices have (in the corresponding minimal cotree) a common parent which is an NTT–vertex of $\oplus$–type (resp. of $\otimes$–type). It is also interesting to note that in the complement of G (i.e. $\overline{G}$), the roles of duplicate and coduplicate are exchanged. In addition, in complements the corresponding trees are obtained by exchanging the types of $\oplus$ and $\otimes$ vertices.*

It is also worth mentioning that any pair of duplicate (resp. coduplicate) vertices gives rise to an eigenvector of G for 0 (resp. -1) defined as follows: all its entries are zero except those corresponding to u and v which can be taken to be 1 and -1, or vice versa. Thus any collection with k mutually duplicate (resp. coduplicate) vertices gives rise to $k-1$ linearly independent eigenvectors for 0 (resp. -1).

We now focus our attention to the polynomial reconstruction problem. Since

$$P'_G(x) = \sum_{i=1}^{n} P_{G_i}(x)$$

(see, for example, [3] p. 60) we can readily determine the characteristic polynomial except for the constant term. If we know any eigenvalue of G, then the constant term is uniquely determined (see [4]). In particular, this will be the case if some polynomial from the polynomial deck has a multiple root. Then, by the Interlacing Theorem, the same root appears in the characteristic polynomial. More generally, if we know the value of the characteristic polynomial in some point, we are again done.

No example of non–unique reconstruction (for $n > 2$) of the characteristic polynomial is known so far. On the other hand, the uniqueness of the polynomial reconstruction is proved for several classes of graphs, like regular graphs [4], trees [5] (see also [2]), unicyclic graphs [16], graphs whose vertex–deleted subgraphs have spectra bounded from bellow by -2 [13, 15], small graphs up to 10 vertices [5], etc. There are also many results proved on reconstructing the bipartite graphs [4], disconnected graphs [5, 11, 16], graphs with terminal vertices [11, 12, 16], etc. It is also worth mentioning that the characteristic polynomial of any graph G (with $n > 2$) is reconstructible if the decks $\mathcal{P}(G)$ and $\mathcal{P}(\overline{G})$ are known (see [9]).

It is known that, from the polynomial deck, the number of vertices $n(G)$, the number of edges $e(G)$, as well as the vertex–degree sequence $(\deg(v_1), \deg(v_2), ..., \deg(v_n))$ are reconstructible. A larger list of invariants and properties which are reconstructible from the polynomial deck one can find in [2] or [16]. For certain classes of graphs, the knowledge of these invariants and/or properties can be sufficient to get a positive answer to Problem 1.

3 The multiplicities of 0 and -1 in the spectra of cographs

The main result of this section is Theorem 3.1. It provides an alternative proof of the results of G. Royle (see [10]) without addressing the characteristic polynomials of graphs G and $\overline{G}$, and answers the question posed in [10] directed to finding such proofs. It is worth mentioning that the first author (T.B.) has also provided a proof (unpublished) of Theorem 3.1, part (i). Both alternative proofs are based on the first cotree (i.e. T_G) for representing cographs.

Let G be a cograph, and A_G its adjacency matrix. Consider a (real) vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ such that $A_G\mathbf{x} = \lambda\mathbf{x}$. Clearly, $\mathbf{x}$ is an eigenvector of G for λ if $\mathbf{x} \neq \mathbf{0}$. Let w be an interior vertex of T_G, and let U_w be the set of terminal vertices which are the successors of w. Then the eigenvalue equation, for a vertex $u \in U_w$, can be written in the following form:

$$\lambda x_u = \sum_{v\in\Gamma(u)\cap U_w} x_v + \sum_{v\in\Gamma(u)\setminus U_w} x_v. \tag{1}$$

For short, we put $S_{u,w} = \sum_{v\in\Gamma(u)\cap U_w} x_v$, and $R_w = \sum_{v\in\Gamma(u)\setminus U_w} x_v$ (note,

R_w does not depend on the choice of $u \in U_w$). Recall at this place that two vertices of G are adjacent (non–adjacent) if their least common ancestor in T_G is of $\otimes$-type (resp. $\oplus$-type).

If w is an NTT–vertex, then

$$S_{u,w} = \begin{cases} 0, & \text{if } w \text{ } is \text{ } a \ \oplus -\text{v}ertex, \\ \sum_{v\in\Gamma(u)\cap U_w} x_v, & \text{if } w \text{ } is \text{ } a \ \otimes -\text{v}ertex. \end{cases}$$

Otherwise, if w is not an NTT–vertex, let $w_1, w_2, \ldots, w_q$ be the children of w in T_G. Assume also that $u \in U_{w_k}$ for some fixed k $(1 \le k \le q)$. Then

$$S_{u,w} = \begin{cases} \sum_{v\in\Gamma(u)\cap U_{w_k}} x_v, & \text{if } w \text{ } is \text{ } a \ \oplus -\text{v}ertex, \\ \sum_{v\in\Gamma(u)\cap U_{w_k}} x_v + \sum_{v\in U_w\setminus U_{w_k}} x_v, & \text{if } w \text{ } is \text{ } a \ \otimes -\text{v}ertex. \end{cases}$$

Let $S_w = \sum_{v\in U_w} x_v$. Then the previous equation becomes

$$S_{u,w} = \begin{cases} \sum_{v\in\Gamma(u)\cap U_{w_k}} x_v, & \text{if } w \ \ is \ a \ \oplus -\text{v}ertex, \\ -\sum_{v\in U_{w_k}\setminus\Gamma(u)} x_v + S_w, & \text{if } w \ \ is \ a \ \otimes -\text{v}ertex. \end{cases}$$

Let $u \in U_w$. If w is an NTT–vertex then (1) can be written as follows:

$$\lambda x_u = R_w, \ \ \text{if } w \text{ } is \text{ } a \ \oplus -\text{v}ertex, \ or \tag{2}$$

$$\lambda x_u = -x_u + S_w + R_w, \ \ \text{if } w \text{ } is \text{ } a \ \otimes -\text{v}ertex. \tag{3}$$

Otherwise, if w is not an NTT–vertex, and if $u \in U_{w_k} \subseteq U_w$ for some k $(1 \le k \le q)$, then (1) can be written as follows:

$$\lambda x_u = \sum_{v\in\Gamma(u)\cap U_{w_k}} x_v + R_w, \ \ \text{if } w \text{ } is \text{ } a \ \oplus -\text{v}ertex, \ or \tag{4}$$

$$\lambda x_u = -\sum_{v\in U_{w_k}\setminus\Gamma(u)} x_v + S_w + R_w, \ \ \text{if } w \text{ } is \text{ } a \ \otimes -\text{v}ertex. \tag{5}$$

Denote by $Sp(G)$ the spectrum of a graph G. The main result in this section reads.

Theorem 3.1 *Let G be a connected cograph on at least two vertices. Then the following holds:*

(i) if G has no duplicate vertices then $0 \notin Sp(G)$;

(ii) if G has no coduplicate vertices then $-1 \notin Sp(G)$.

Proof Observe first that r, the root of T_G, is the vertex of $\otimes$–type (note, otherwise G is disconnected). Consider next NTT–vertices, and let w be any of them. Then w has only one child provided $\lambda = 0$ and it is of $\oplus$–type, or $\lambda = -1$ and it is of $\otimes$–type (otherwise, G has duplicate or coduplicate vertices, respectively).

To prove the theorem, we will show that the vector $\mathbf{x}$, satisfying $A_G\mathbf{x} = \lambda\mathbf{x}$ (for $\lambda \in \{0, -1\}$), is non–negative, or non–positive. But this is a contradiction (since $\mathbf{x}$ is not orthogonal to the eigenspace corresponding to the largest eigenvalue of G, which is a connected graph).

Our proof is based on induction arguments.

Induction basis: Assume that w is a fixed NTT–vertex. We claim that all x_u's with $u \in U_w$ are ≥ 0, or ≤ 0 (for $\lambda \in \{0, -1\}$). Clearly, we can take further on that $|U_w| > 1$.

First, for $\lambda = 0$, w must be of $\otimes$–type (otherwise, G has duplicate vertices). So, by (3), $x_u = S_w + R_w$ for any $u \in U_w$. Since S_w and R_w depend only on w, x_u is constant for $u \in U_w$, and we are done. Secondly, for $\lambda = -1$, w must be of $\oplus$–type (otherwise, G has coduplicate vertices). So, by (2), $x_u = -R_w$ for any $u \in U_w$, and we are again done.

Induction hypothesis: Assume now that w is a fixed interior vertex of T_G, but not an NTT–vertex. Let $w_1, w_2, \ldots, w_q$ be its children. Clearly, we can take that $q > 1$. Assume next that the following condition holds: for fixed k $(1 \leq k \leq q)$, all x_u's with $u \in U_{w_k}$ are ≥ 0, or ≤ 0. At this place we can say that the children of w are scanned, while w has to be scanned. (So, in the previous part of the proof, NTT–vertices were scanned; now we are scanning w.) Note also that a vertex such as w above always exists (it can be encountered by moving from the root downwards to NTT–vertices).

Induction step: We need now to prove that all x_u's with $u \in U_w$ are ≥ 0, or ≤ 0. For this aim, assume to the contrary, and let u_s and u_t be the vertices of U_w, for which, say $x_{u_s} > 0$, while $x_{u_t} < 0$. Then, $u_s \in U_{w_i}$, while $u_t \in U_{w_j}$ for some fixed $i \neq j$ $(1 \leq i, j \leq q)$.

We now distinguish two cases depending on λ.

Case 1: $\lambda = 0$. First, let w be a vertex of $\oplus$–type. Then $x_v \geq 0$ for $v \in U_{w_i}$ (by induction hypothesis, since $u_s \in U_{w_i}$). So, by (4), $R_w \leq 0$ (note, x_{u_s} is not included in corresponding sum). Similarly, $x_v \leq 0$ for $v \in U_{w_j}$ (by induction hypothesis, since $u_t \in U_{w_j}$). So, by (4), $R_w \geq 0$. Consequently, $R_w = 0$. In addition, since w_k $(1 \leq k \leq q)$ is of $\otimes$–type, $R_{w_k} = R_w$, and thus $R_{w_k} = 0$. Consider next G_{w_k}. It does not have duplicate vertices (otherwise, $G = G_r$ would have), and is connected (since w_k is of $\otimes$–type). Let $\mathbf{y}_k$ be a restriction of $\mathbf{x}$ on U_{w_k}. Clearly, $A_{G_{w_k}}\mathbf{y}_k = \lambda\mathbf{y}_k$ and $\mathbf{y}_k \geq \mathbf{0}$, or $\leq \mathbf{0}$. For $k = i$ (or $k = j$) $\mathbf{y}_k \neq \mathbf{0}$, since u_s (or u_t) belongs to U_{w_k}, and therefore x_{u_s} (or x_{u_t}) is an entry of $\mathbf{y}_k$. But this is a contradiction as desired (with $\mathbf{y}_k$ and G_{w_k} in the role of $\mathbf{x}$ and G).

Secondly, let w be a vertex of $\otimes$–type. By putting in (5) $k = i$, $u = u_s$, we get $S_w + R_w > 0$ (note, that in this case (5) can be written as $S_w + R_w = \sum_{v \in U_{w_i} \setminus \Gamma(u_s)}$, and x_{u_s} is included in corresponding sum). Similarly, by putting in (5) $k = j$, $u = u_t$, we get $S_w + R_w < 0$, a contradiction.

So we have encountered a contradiction at some intermediate stage, or have proved an induction step; consequently, the proof of part (i) follows.

Case 2: $\lambda = -1$. First, let w be a vertex of $\oplus$–type. Then (4) can be written in the following form

$$\sum_{v \in \Gamma[u] \cap U_{w_k}} x_v + R_w = 0.$$

Taking that $k = i$, $u = u_s$, and also that $k = j$, $u = u_t$, we get that $R_w < 0$, and respectively $R_w > 0$, a contradiction.

Secondly, let w be a vertex of $\otimes$–type. Then (5) can be written in the following form

$$-\sum_{v \in \Gamma[u] \setminus U_{w_k}} x_v + S_w + R_w = 0.$$

Similarly as in Case 1 (subcase $\lambda = 0$) we get that $S_w + R_w = 0$ (by putting in (5) $k = i$, $u = u_s$, and $k = j$, $u = u_t$). Therefore, by (5),

$$\sum_{v \in U_{w_k} \setminus \Gamma[u]} x_v = 0 \tag{6}$$

for all $u \in U_{w_k}$ ($1 \leq k \leq q$). Henceforth, assume that $k = i$, or $k = j$. Then we first get that all pairs of vertices u_1 and u_2 from U_{w_k} for which x_{u_1} and x_{u_2} are not both zero are adjacent (otherwise (6) fails to hold). Consider now the vertices u_s and u_t (as chosen above). They are mutually adjacent, and also adjacent to all vertices from $(U_{w_i} \cup U_{w_j}) \setminus \{u_s, u_t\}$ (as pointed above). In addition, they have the same neighbours out of U_w, and so u_s and u_t are coduplicate in G, a contradiction.

So we have encountered a contradiction at some intermediate stage, or have proved an induction step; consequently, the proof of part (ii) follows.

This completes the proof. □

We will now deduce several consequences of the above result. Let $m(\lambda; G)$ denote the multiplicity of an eigenvalue λ of G. We will now assume that the corresponding cotree of a cograph G is a minimal cotree $\widehat{T}_G$.

Corollary 3.1 *If G is a cograph then $m(0; G) + m(-1; G) \geq 1$.*

Proof The statement is trivial if G has only one vertex. If G has at least two vertices then there is an NTT–vertex in $\widehat{T}_G$ with at least two children and therefore, depending on its type, either $m(0;G) \geq 1$ or $m(-1;G) \geq 1$, and the proof follows. □

Corollary 3.2 *If G is a cograph which does not contain isolated vertices, then*

(i) $m(0;G) = \sum_{w \in V_0}(t_w - 1)$;

(ii) $m(-1;G) = \sum_{w \in V_1}(t_w - 1)$,

where V_0 (resp. V_1) is the set of interior vertices (in $\widehat{T}_G$) of $\oplus$–type (resp. $\otimes$–type) having t_w children as terminal vertices. In addition, 0 *and* −1 *are the non–main eigenvalues of G.*

Proof We first prove that two vertices, say u and v, are duplicate (resp. coduplicate) in G if they have a common parent in $\widehat{T}_G$. Assume (for contradiction) that w is a common ancestor of u and v, but not a common parent. If so, there exists a vertex w' of the type opposite to the type of w, and belonging to w–u (or w–v) path in $\widehat{T}_G$. Let c be a terminal successor of w' (note w' has at least two children). But then c is adjacent to exactly one of the vertices u and v, and consequently, they are neither duplicate, nor coduplicate. Therefore, w must be a parent of u and v.

Taking into consideration the positions of all collections of mutually duplicate (coduplicate) vertices of G, we immediately get (i) and (ii) (see also Remark 2.1).

This completes the proof. □

Remark 3.1 *To compute all other eigenvalues of G (being a cograph) we can make use of the divisor technique (see, for example, [3], Chapter 4). To get an equitable partition, we can take that each cell is either a collection of mutually duplicate or coduplicate vertices, or it is a singleton consisting of the remaining vertices. In this situation we have*

$$P_G(x) = x^{m(0;G)}(x+1)^{m(-1;G)}D_G(x),$$

where $m(0;G)$ and $m(-1;G)$ are given in Corollary 3.2, while $D_G(x)$ is the characteristic polynomial of the divisor of G (as specified above).

4 A positive result in the polynomial reconstruction problem

In this section we will consider the polynomial reconstruction problem for graphs whose polynomial deck consists only of σ–graphs. Recall first that any such graph G on at least 5 vertices has no (induced) subgraphs equal to $2K_2$; otherwise, $\lambda_2(G-v) \geq 1$ for some $v \in V_G$ (by the Interlacing Theorem). (Note also that the polynomial reconstruction problem is resolved for small graphs up to 10 vertices.) So all components of G but one are isolated vertices, if G is to be disconnected. On the other hand (see [5], Theorem 8 and Corollary 1), if G as a disconnected graph is a counterexample to the reconstruction problem, it must have just two components of the same order. Therefore, we can assume further on that G is connected. Similarly, we can conclude that G does not contain P_5 as an induced subgraph.

First, we have the following lemma.

Lemma 4.1 *A σ–graph which is not a cograph is a σ^0–graph.*

Proof Recall that the second largest eigenvalue of any σ–graph does not exceed σ. On the other hand, each graph which is not a cograph contains a path P_4 as an induced subgraph, and therefore its second largest eigenvalue is bounded from below by σ.

This completes the proof. □

In the remainder of the section we will distinguish two cases:

(i) at least one vertex–deleted subgraph of G contains P_4 as an induced subgraph, and

(ii) none of vertex–deleted subgraphs of G contains P_4 as an induced subgraph.

We resolve the case (i) in the following theorem.

Theorem 4.1 *If the deck of G consists of σ–graphs and if P_4 is an induced subgraph of $G' = G - v$ for some v, then the polynomial reconstruction is unique.*

Proof Observe first that G' must be a σ^0–graph (by assumptions and by the Interlacing Theorem). So, any vertex in G', say u, should be of these types:

(1) non–adjacent to a vertex of P_4,

(2) adjacent to both terminal vertices of P_4,

(3) adjacent to both non–terminal vertices of P_4, and

(4) adjacent to all vertices of P_4

(see [7], Corollary 2.3). The same holds for the vertex v (by exchanging the roles of u and v). Therefore, each vertex of G, outside of P_4 has one of the above types. Therefore, we get that G contains σ as an eigenvalue (not necessarily the second one). To see this, define an eigenvector (for σ) as follows: let 1, σ, $-\sigma$ and -1 be the entries corresponding to vertices of P_4 (in natural order); let all other entries be equal to 0. In this way, we get that σ is an eigenvalue of G, and so the polynomial reconstruction is unique.

This completes the proof. □

Before we consider the case (ii) we need the following two lemmas.

Lemma 4.2 *Let H be a cograph such that $\lambda_2(H) \leq \sigma$, and let r be the root of $\widehat{T}_H$. If r is of $\otimes$–type (resp. $\oplus$–type) then each terminal vertex of $\widehat{T}_H$ is at distance at most 7 (resp. 8) from the root.*

Proof First, let r be of $\otimes$–type. Assume to the contrary that there is a terminal vertex in $\widehat{T}_H$ at distance 8 from r. If so, the tree of Fig. 2 appears in $\widehat{T}_H$ as an induced subtree, and let H' be the corresponding cograph (it is a an induced subgraph of H). Since $\lambda_2(H') > \sigma$ (a computational argument) we get a contradiction (by the Interlacing Theorem).

Secondly, let r be of $\oplus$–type. So its children are the vertices of $\otimes$–type (or terminal vertices). Since the distance between these vertices and their terminal successors cannot exceed 7 (in virtue of the above arguments), the distance between r and terminal vertices cannot exceed 8, and we are done.

This completes the proof. □

Lemma 4.3 *Let H be a cograph on at least 10 vertices satisfying $\lambda_2(H) \leq \sigma$. Then $m(0; H) + m(-1; H) \geq 2$.*

Proof By the previous lemma, each terminal vertex in $\widehat{T}_H$ is at distance at most 8 from the root (of $\widehat{T}_H$). Since H has at least 10 vertices we get that at least one interior vertex (of $\widehat{T}_H$) has at least three children as terminal vertices, or at least two interior vertices have two children as terminal vertices and the proof follows. □

Remark 4.1 *Regarding to the formulation of the previous lemmas, it is noteworthy to add that the following question (posed by the second author) is still open: Is there a cograph whose second largest eigenvalue is equal to σ?*

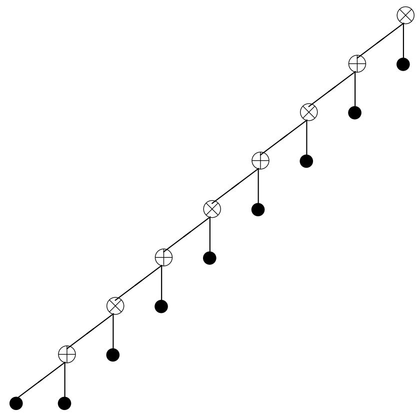

Fig. 2

We are now in position to prove the next theorem.

Theorem 4.2 *If the deck of G consists of σ–graphs none of them containing P_4 as an induced subgraph, then the polynomial reconstruction is unique.*

Proof Clearly, we can assume that G has at least 11 vertices. If so, G does not contain P_4 as an induced subgraph (since otherwise, $G' = G - v$ for some v would contain P_4, contrary to assumptions). Therefore, G is a cograph. Since every vertex–deleted subgraph of G has at least 10 vertices we have (by Lemma 4.3) that $m(0; G') + m(-1; G') \geq 2$ for every such subgraph.

Now, we distinguish two cases:

Case 1: for at least one subgraph $G' = G - v$ ($v \in V(G)$), $m(0; G') \geq 2$ or $m(-1; G') \geq 2$. Then, by the Interlacing Theorem, G contains 0 or -1 as an eigenvalue, and so the polynomial reconstruction is unique.

Case 2: for all subgraphs $G' = G - v$ ($v \in V(G)$), we have $m(0; G') = m(-1; G') = 1$. Then we claim that both numbers, 0 and -1, are the eigenvalues G. First, we have (as above) that $m(0; G) + m(-1; G) \geq 2$. So, in the worst case we can have either $m(0; G) = 2$ and $m(-1; G) = 0$, or $m(0; G) = 0$ and $m(-1; G) = 2$. But then we have in G either two pairs, or one triplet, of mutually duplicate (resp. coduplicate) vertices. But then, by deleting a vertex (say v) not in one of the above sets, we get the same situation in $G' = G - v$, a contradiction. This proves our claim, and so the polynomial reconstruction is unique.

This completes the proof. □

Collecting the results above we arrive at the main result of this section.

Theorem 4.3 *The polynomial reconstruction is unique for those graphs whose vertex–deleted subgraphs have the second largest eigenvalue not exceeding* $\frac{\sqrt{5}-1}{2}$.

Proof If there is a graph in the deck which contains P_4 as an induced subgraph, the polynomial reconstruction is unique by Theorem 4.1. Otherwise, the polynomial reconstruction is unique by Theorem 4.2.

This completes the proof. □

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