Desanka Radunović – NUMERIČKE METODE

Aproksimacija

Aproksimacija u HILBERTOVOM prostoru

norma $\|f\| = \sqrt{(f,f)}$, rastojanje $\|f-g\|^2 = (f-g,\ f-g)$

Element najbolje aproksimacije $Q_0 = \sum_{i=1}^{n} c_i^\circ\, g_i$

$$\big(E_n(f)\big)^2 = \left\| f - \sum_{i=1}^{n} c_i^\circ\, g_i \right\|^2 = \left(f - \sum_{i=1}^{n} c_i^\circ\, g_i\,,\ f - \sum_{i=1}^{n} c_i^\circ\, g_i \right)$$

POSTOJI, jer je prostor linearan i normiran,

JEDINSTVEN JE, jer je prostor strogo normiran

♠ $$E_n(f) = \|f - Q_0\| \quad \Longleftrightarrow \quad (f - Q_0,\, Q) = 0, \quad \forall\, Q = \sum_{i=1}^{n} c_i\, g_i$$

Kako odrediti element najbolje aproksimacije za f ?

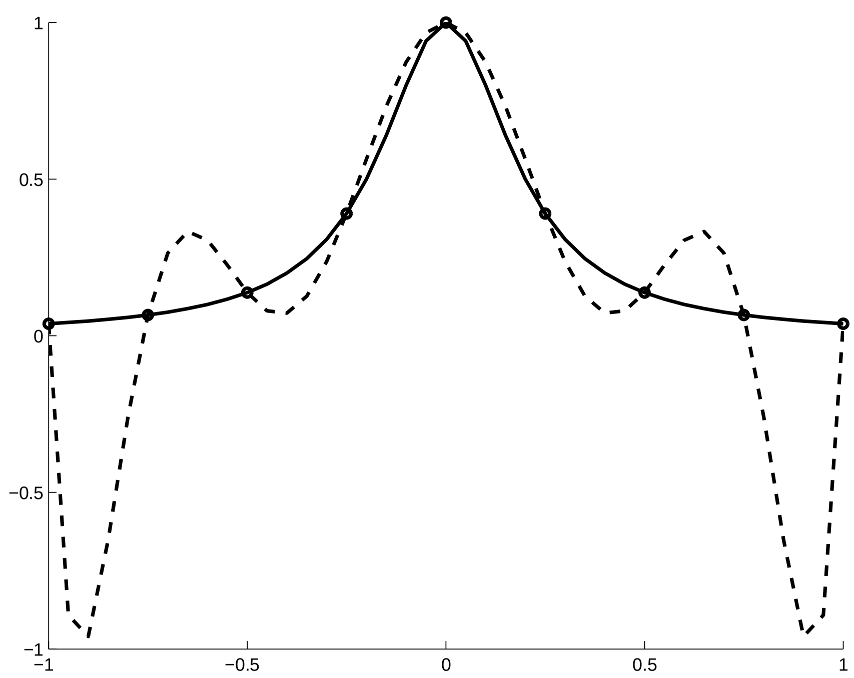
$$Q = g_j, \quad (f - Q_0,\, g_j) = 0, \qquad \boxed{\sum_{i=1}^{n} c_i^{\circ}(g_i, g_j) = (f, g_j)} \quad j = 1, \ldots, n,$$

$$\big(g_i\,,\, g_j\big) = \delta_{ij}, \qquad c_j^{\circ} = \big(f\,,\, g_j\big)$$

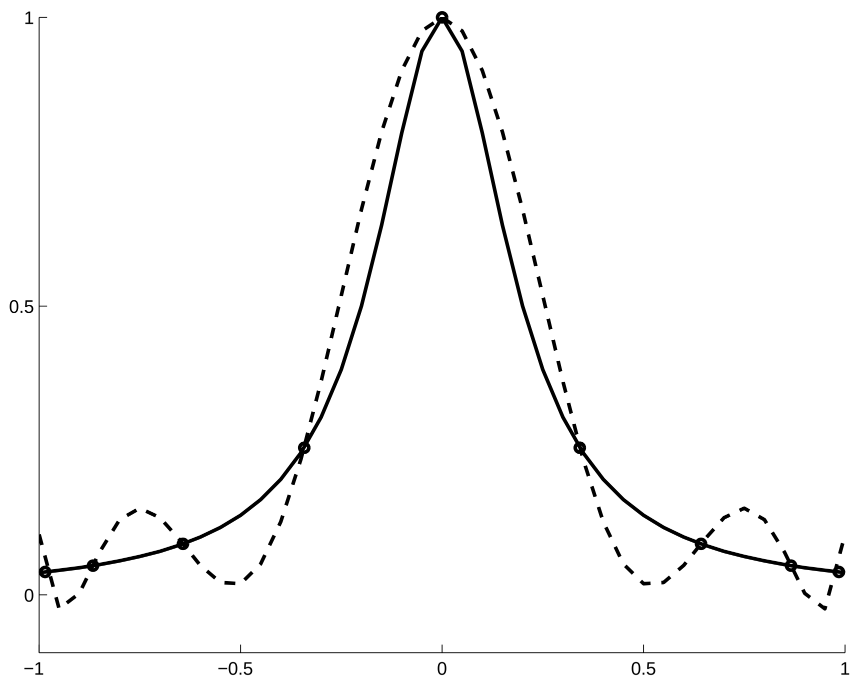
$$Q_{\circ} = \sum_{i=1}^{n} (f, g_i)\, g_i, \qquad \big(E_n(f)\big)^2 = \|f - Q_{\circ}\|^2 = \|f\|^2 - \sum_{i=1}^{n} |(f, g_i)|^2$$

Bessel-ova nejednakost, Parseval-ova jednakost ($n = \infty$)

◇ Lagrange-ova interpolacija funkcije $\frac{1}{1+25x^2}$ sa čvorovima

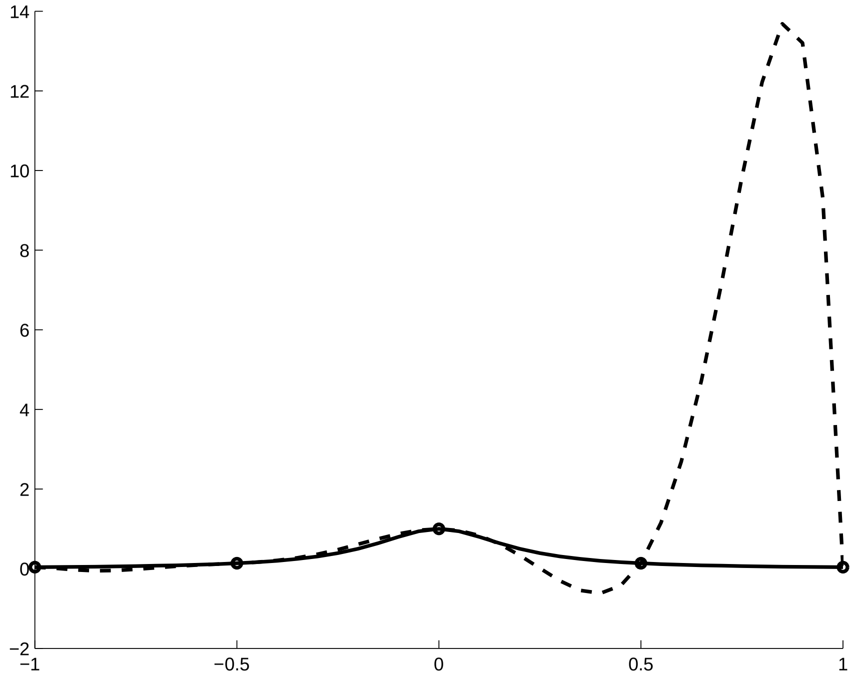


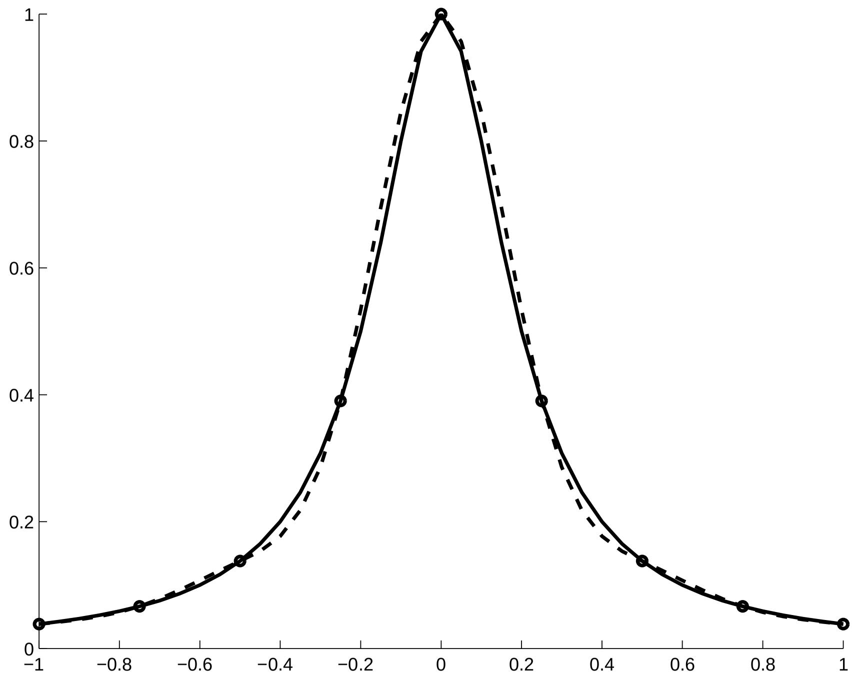
ravnomerno rasporedjenim,



Čebišev-ljevim

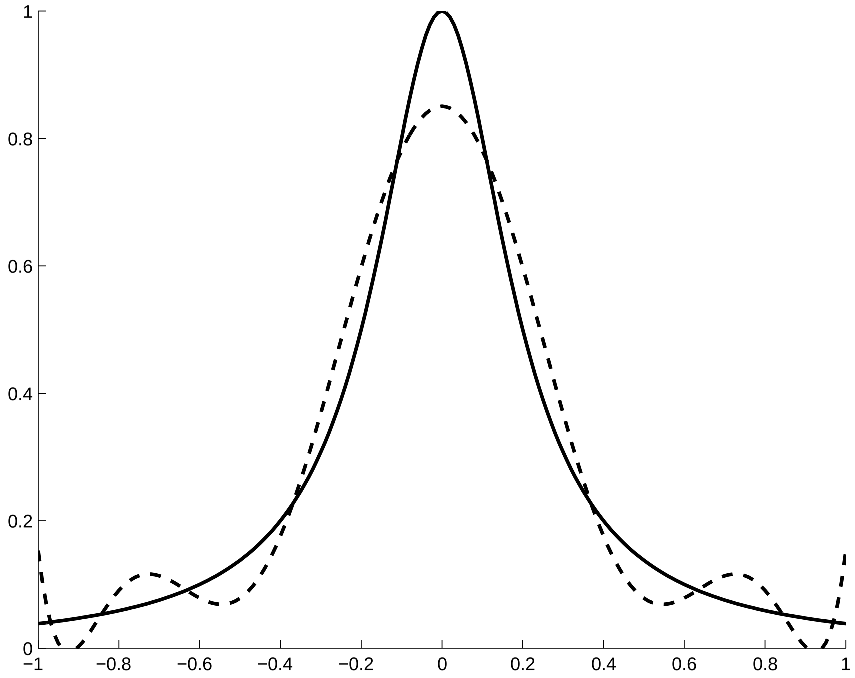
Interpolacija funkcije $\frac{1}{1+25x^2}$



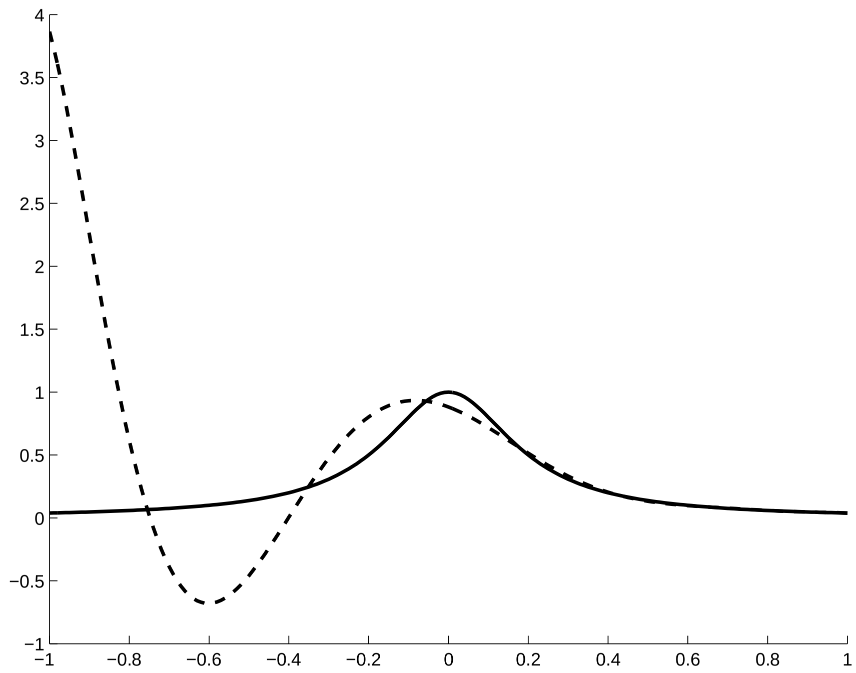


Hermite-ovim polinomom (2, 3, 2, 1, 1), kubnim splajnom

Srednjekvadratna aproksimacija funkcije $\frac{1}{1+25x^2}$ polinomom osmog stepena



sa težinskom funkcijom $p(x) \equiv 1$,



sa težinskom funkcijom $p(x) = e^{10x}$.

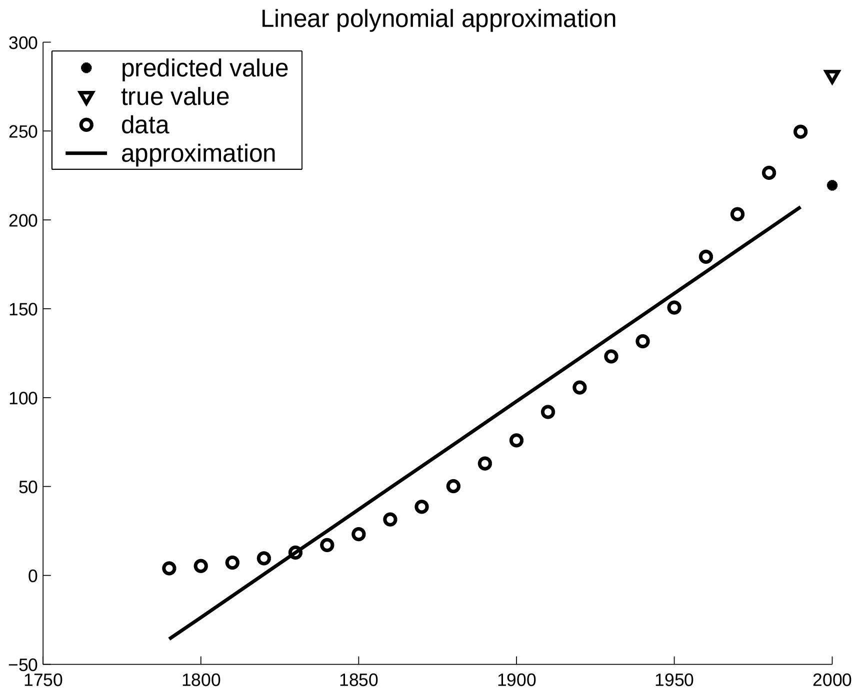
SREDNJEKVADRATNA aproksimacija ($\mathcal{L}_2$)

skalarni proizvod $$(f,\, g) = \int_a^b p(x)\, f(x)\, g(x)\, dx, \qquad p(x) > 0$$

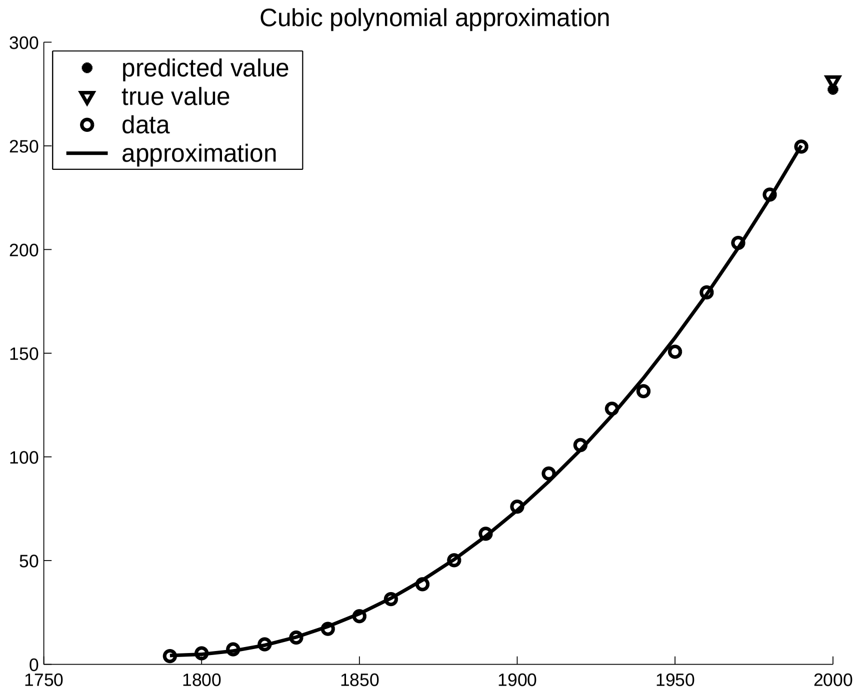
norma $$\|f\|^2 = \int_a^b p(x)\, \big(f(x)\big)^2\, dx$$

$$\big(E_n(f)\big)^2 = \|f - Q_0\|^2 = \inf_Q \int_a^b p(x)\, \big(f(x) - Q(x)\big)^2\, dx$$

◇ Procena broja stanovnika SAD godine 2000. određena diskretnom varijantom srednjekvadratne aproksimacije



polinomom prvog stepena,



polinomom trećeg stepena.

Metoda NAJMANJIH KVADRATA

Gauss (1801-02), procena orbite asteroida Ceres

skalarni proizvod $\quad (f,\, g) = \sum_{i=0}^{n} p_i\, f(x_i)\, g(x_i)\, dx, \qquad p_i > 0$

norma $\quad \|f\|^2 = \sum_{i=0}^{n} p_i \left(f(x_i)\right)^2$

$$\left(E_n(f)\right)^2 = \|f - Q_0\|^2 = \inf_Q \sum_{i=0}^{n} p_i \left(f(x_i) - Q(x_i)\right)^2 dx$$

◇ Procena broja stanovnika SAD 2000. godine pravom

$$P_1(x) = c_0 + c_1 x, \quad (g_j(x) = x^j) \qquad F(c_0, c_1) = \sum_{i=0}^{20} (u(x_i) - c_0 - c_1 x_i)^2$$

$$P_1(x_i) = u(x_i), \quad i = 0, \ldots, 20,$$

$$\frac{\partial F}{\partial c_0} = 0, \qquad \frac{\partial F}{\partial c_1} = 0$$

tj.

$$c_0 \sum_{i=0}^{20} 1 \cdot 1 + c_1 \sum_{i=0}^{20} x_i \cdot 1 = \sum_{i=0}^{20} u(x_i) \cdot 1$$

$$c_0 \sum_{i=0}^{20} 1 \cdot x_i + c_1 \sum_{i=0}^{20} x_i \cdot x_i = \sum_{i=0}^{20} u(x_i) \cdot x_i$$

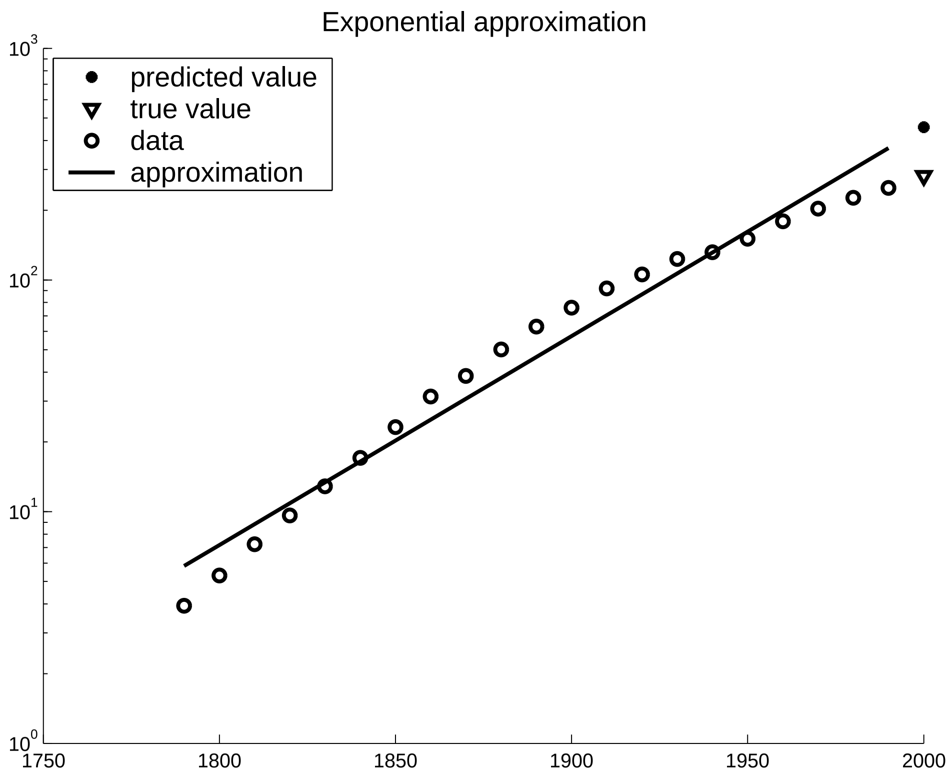
$$A^\top A \mathbf{c} = A^\top \mathbf{b},$$

gde je

$$A = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_{20} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{20}) \end{pmatrix}$$

Procena eksponencijalnom funkcijom

$$u(x) \approx Q(x) = \bar{c}_0 e^{c_1 x}$$

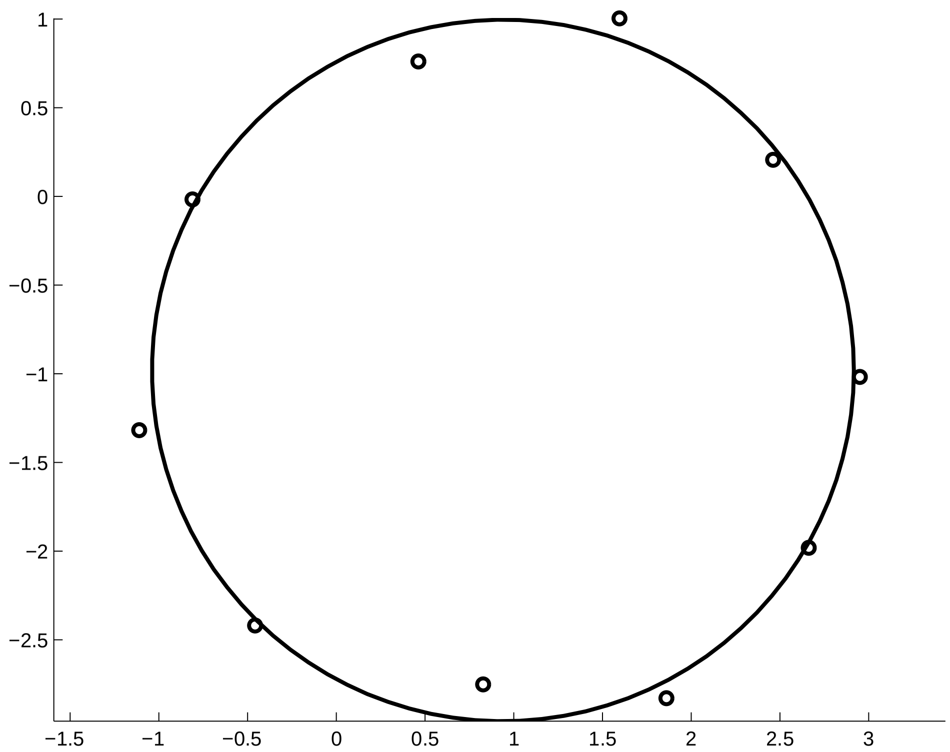


$$F(\bar{c}_0, c_1) = \sum_{i=0}^{20} \big(u(x_i) - \bar{c}_0\, e^{c_1 x_i}\big)^2$$

$$\begin{aligned} \ln\big(u(x)\big) &\approx \ln\big(Q(x)\big) \\ &= \ln(\bar{c}_0) + c_1 x \\ &= c_0 + c_1 x \end{aligned}$$

◇ Aproksimacija tačaka kružnicom

$$(x - c_1)^2 + (y - c_2)^2 = r^2$$



nelinearan problem

$$F(c_1, c_2, r) = \sum_{i=1}^{n} \left(r^2 - (x_i - c_1)^2 - (y_i - c_2)^2\right)^2$$

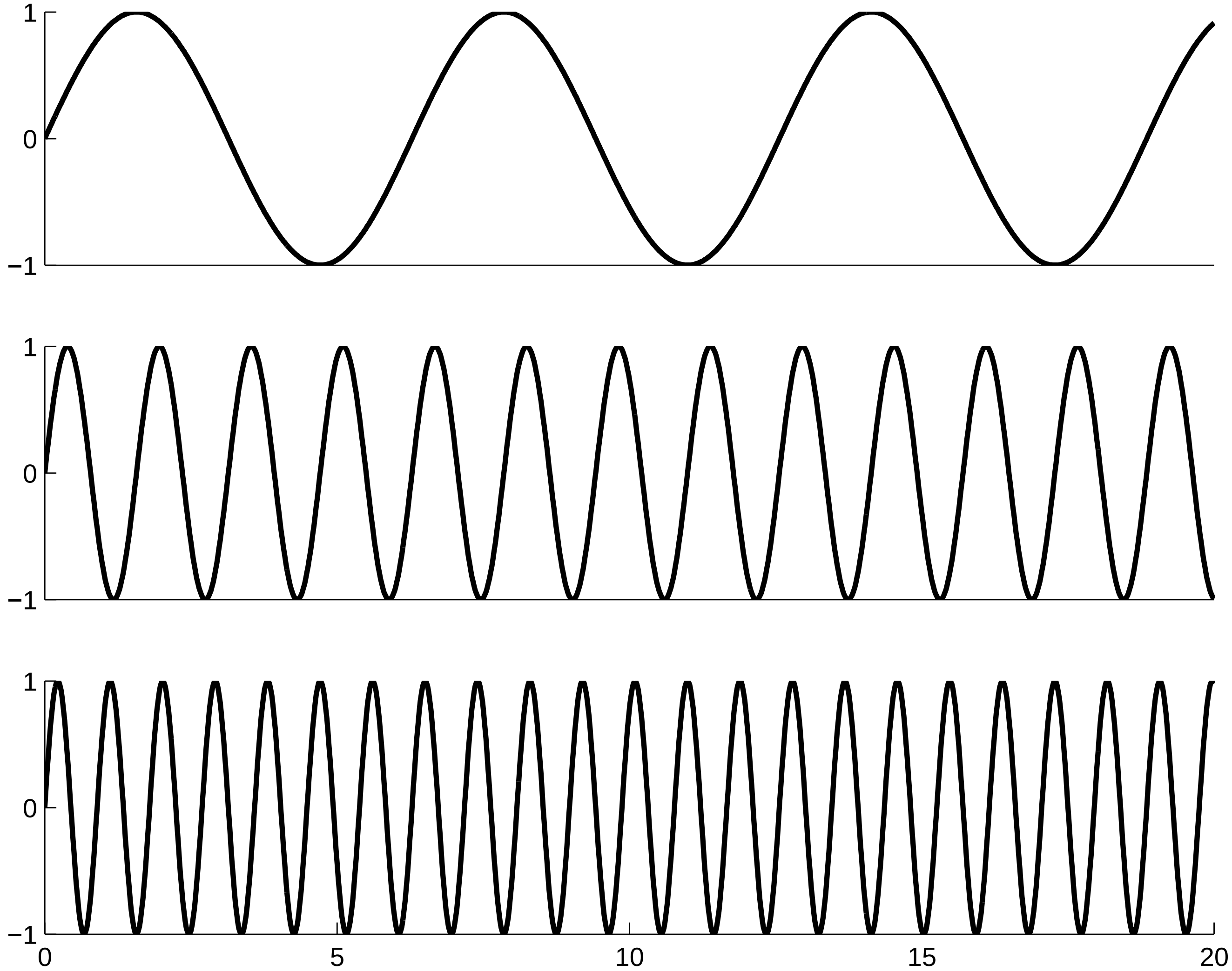
linearan problem

$$c_3 = r^2 - c_1^2 - c_2^2$$

$$2x_i c_1 + 2y_i c_2 + c_3 = x_i^2 + y_i^2$$

$$A^\top A\,\mathbf{c} = A^\top \mathbf{b}$$

Harmonici $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx, \ldots$



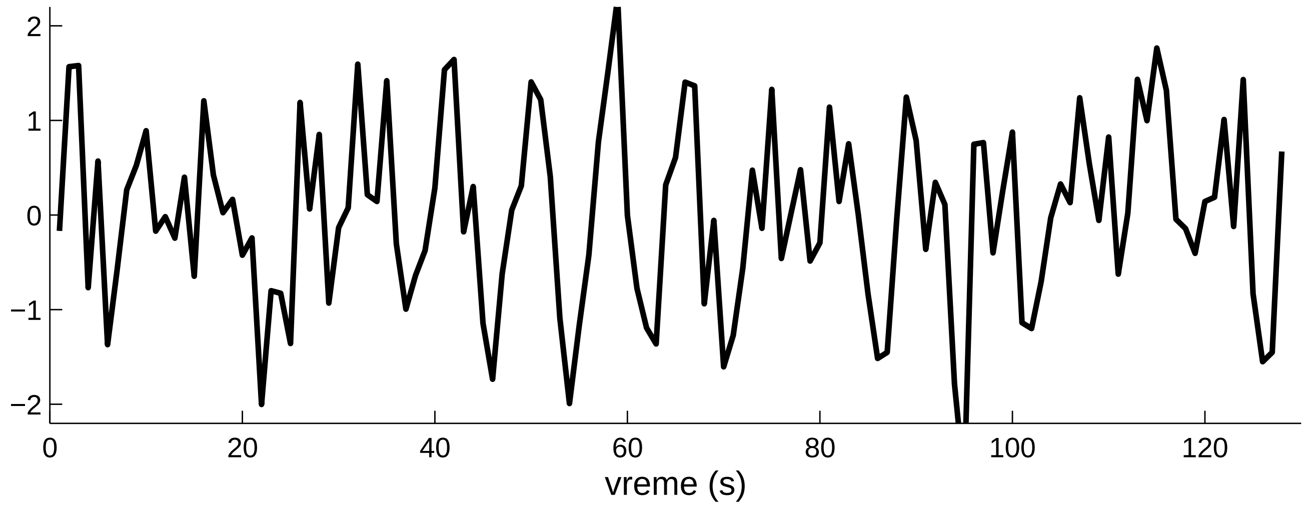
FOURIER-ova ANALIZA (Joseph Fourier,1807)

$$Q(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

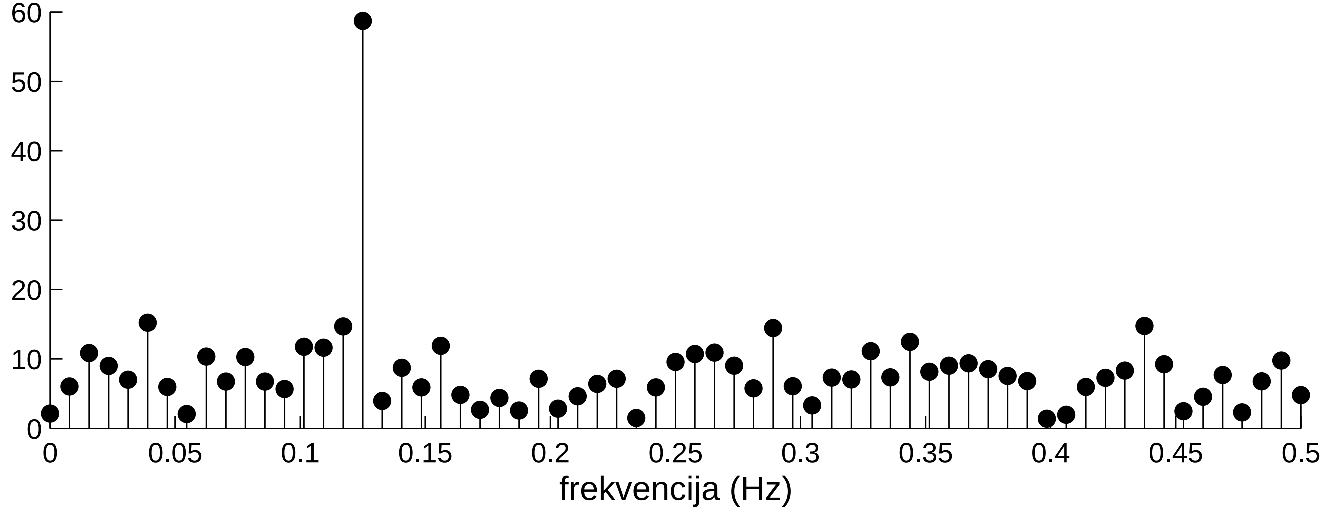
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} \left(f - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right)^2 dx = 0.$$

$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} c_k e^{-\imath kx}, \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{\imath kx} \, dx}$$

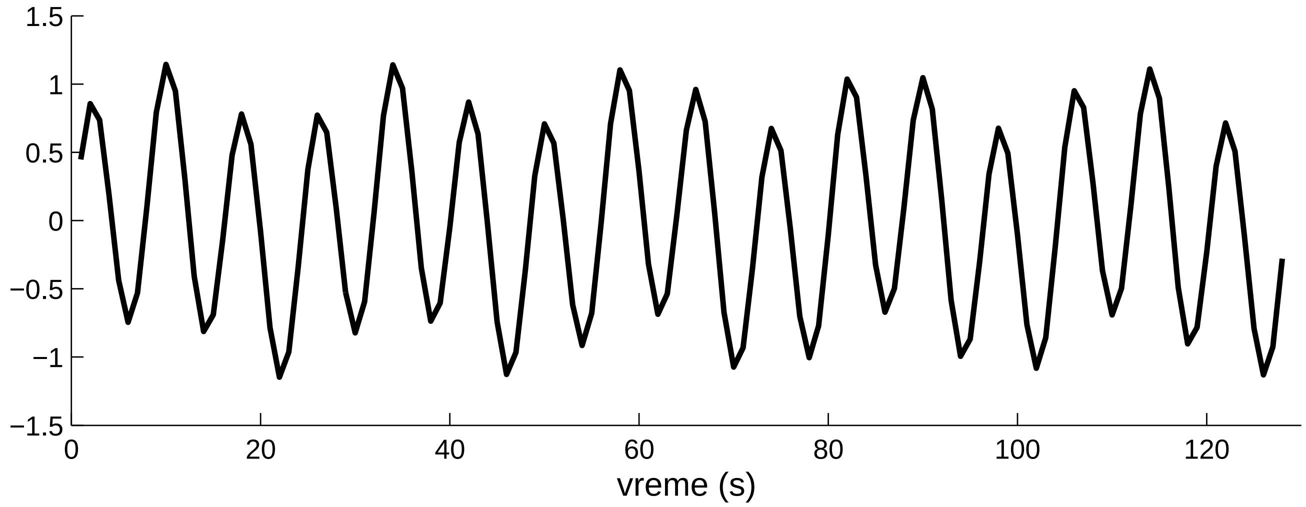


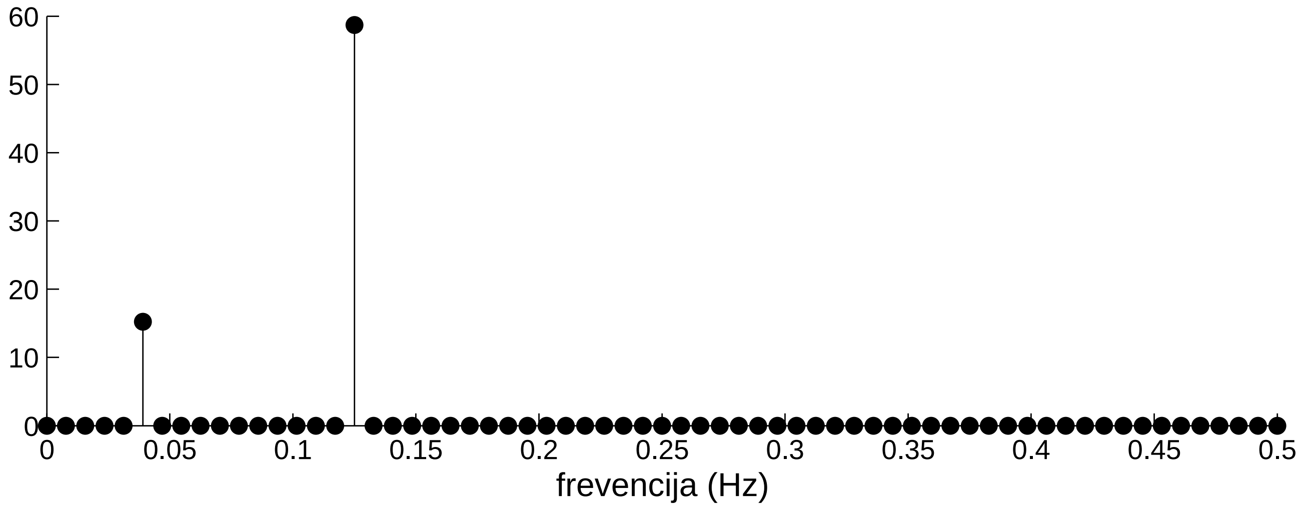
Vremenski
domen signala



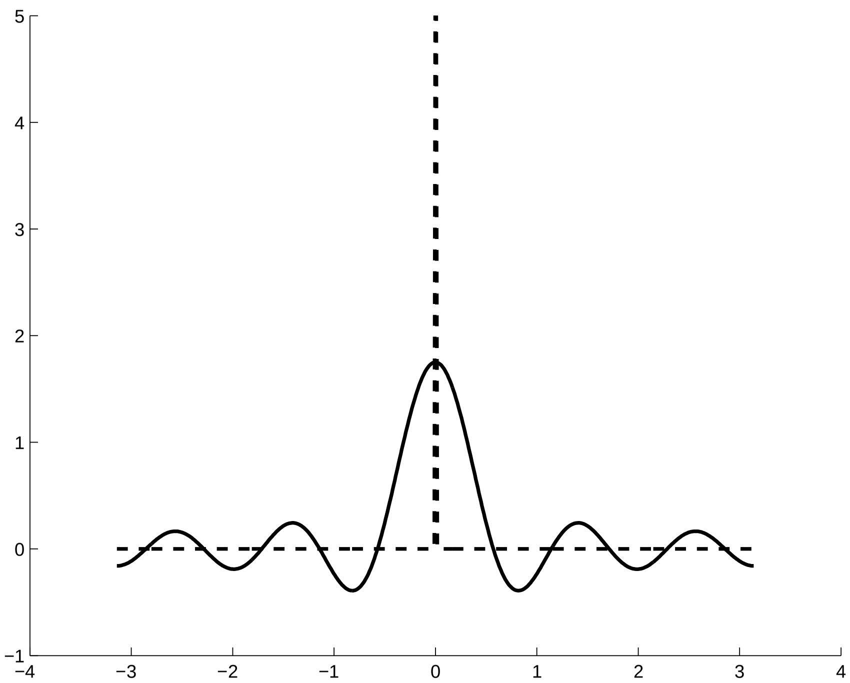
Frekvencijski
domen signala

Kompresija signala u frekvencijskom domenu

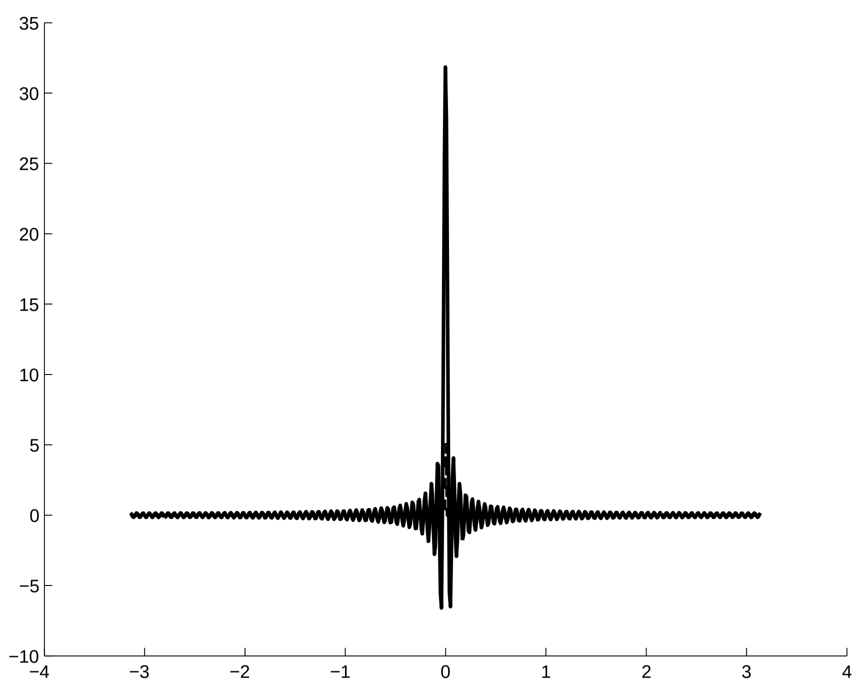




◇ Suma Fourier-ovog reda Dirac-ove funkcije

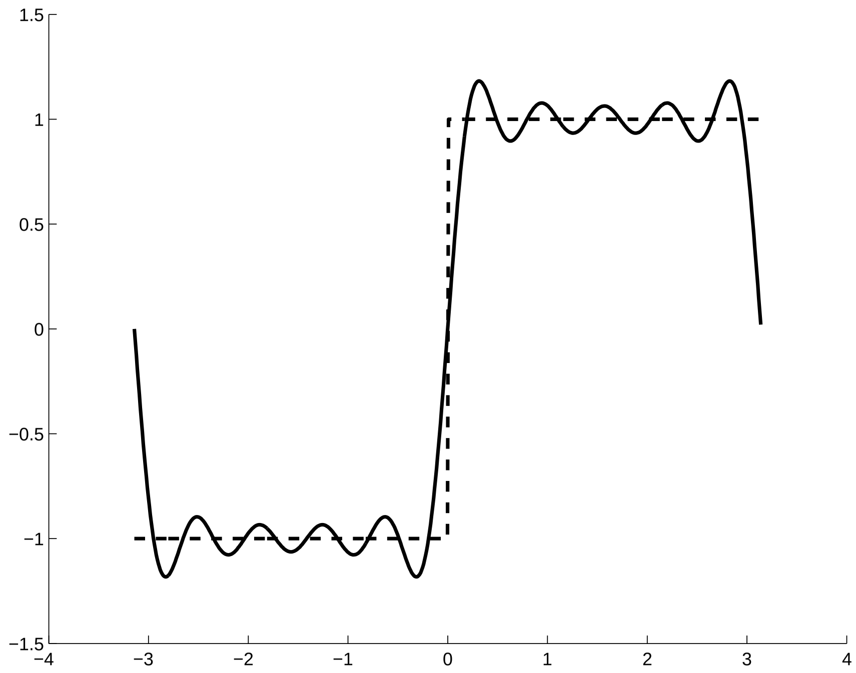


5 sabiraka,



100 sabiraka

◇ Suma Fourier-ovog reda Heaviside-ove funkcije



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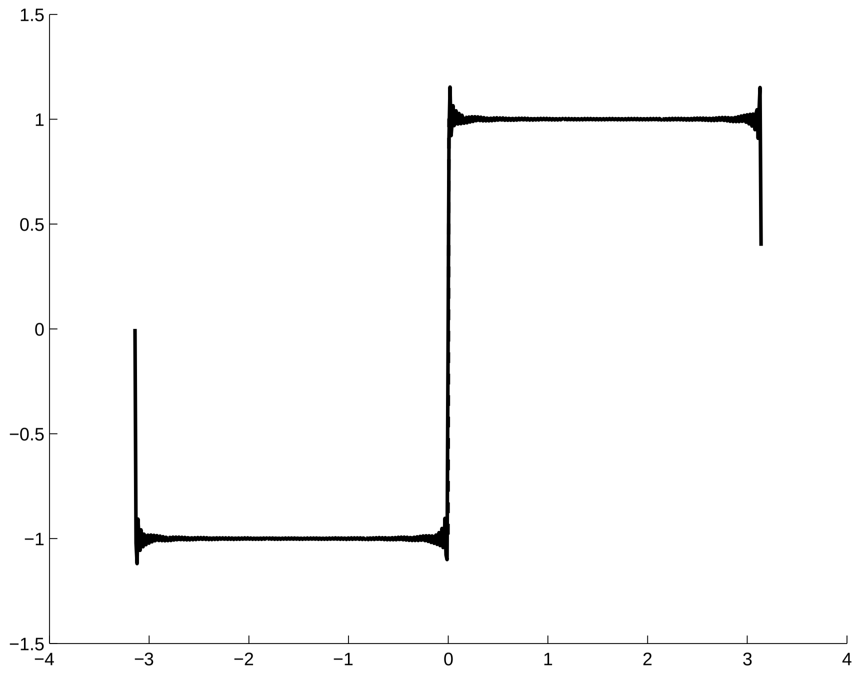
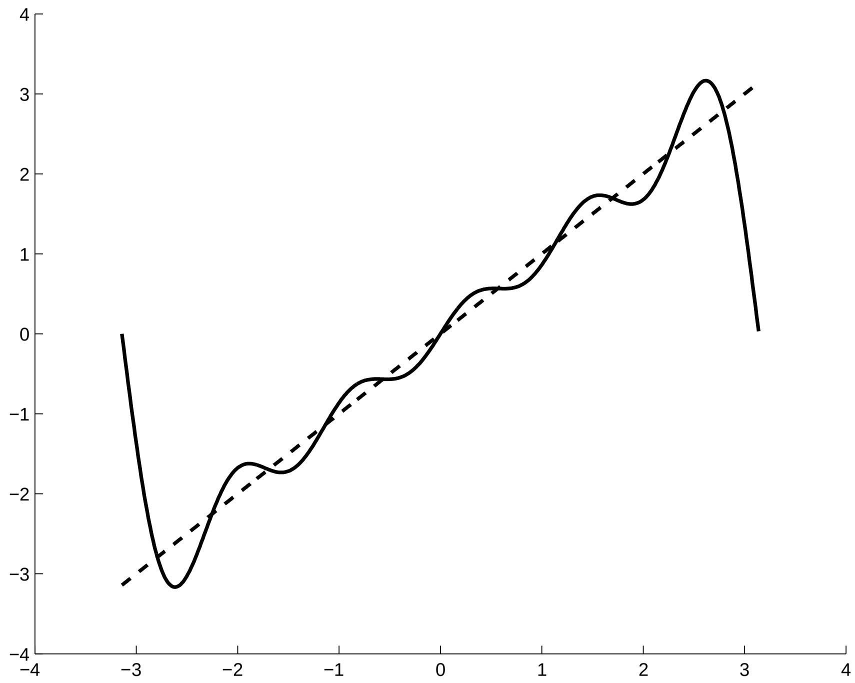
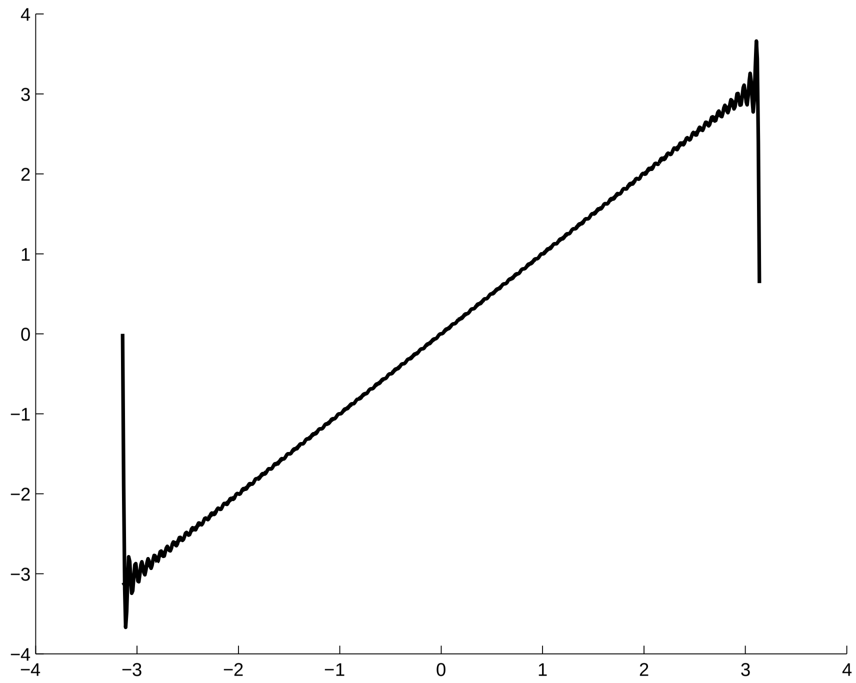


100 sabiraka

- ◇ Suma Fourier-ovog reda linearne funkcije



5 sabiraka,



100 sabiraka

Sopstvene funkcije $\frac{d}{dx}e^{\imath kx} = \imath k\, e^{\imath kx}, \qquad \Delta\, e^{\imath kx} = \left(\frac{e^{\imath kh}-1}{h}\right)\, e^{\imath kx}$

Neperiodična funkcija $x = \frac{2\pi}{T}t, \quad f_T(t) \equiv f(\frac{2\pi}{T}t), \quad K = k\frac{2\pi}{T} = k\Delta K$

$$c_k = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(\tfrac{2\pi}{T}t)e^{\imath k\frac{2\pi}{T}t}\,dt = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t)e^{\imath Kt}\,dt,$$

$$f_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{-\imath Kt} = \sum_{k=-\infty}^{\infty} \frac{1}{T}e^{-\imath Kt}\left(\int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t)e^{\imath Kt}\,dt\right)$$

$T \longrightarrow \infty$

$$\boxed{F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{F}(K)e^{-\imath Kt}\,dK, \qquad \widehat{F}(K) = \int_{-\infty}^{\infty} F(t)e^{\imath Kt}\,dt}$$

Diskretna Fourier-ova transformacija

$$x_j = j\,\frac{2\pi}{n}, \qquad f_j = f(x_j), \quad j = 0, \ldots, n-1, \qquad W = e^{\imath \frac{2\pi}{n}} = \sqrt[n]{e^{\imath 2\pi}}$$

$$\sum_{k=0}^{n-1} c_k \overline{W}^{kj} = f_j, \quad j = 0, \ldots, n-1, \qquad \frac{1}{n} \sum_{j=0}^{n-1} f_j W^{kj} = c_k, \quad k = 0, \ldots, n-1,$$

$$F^* \mathbf{c} = \mathbf{f}, \qquad \frac{1}{n} F\,\mathbf{f} = \mathbf{c}$$

$$(F^* F = n\,I)$$

$$F = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W & W^2 & \ldots & W^{n-1} \\ 1 & W^2 & W^4 & \ldots & W^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{n-1} & W^{2(n-1)} & \ldots & W^{(n-1)^2} \end{pmatrix}$$

FFT

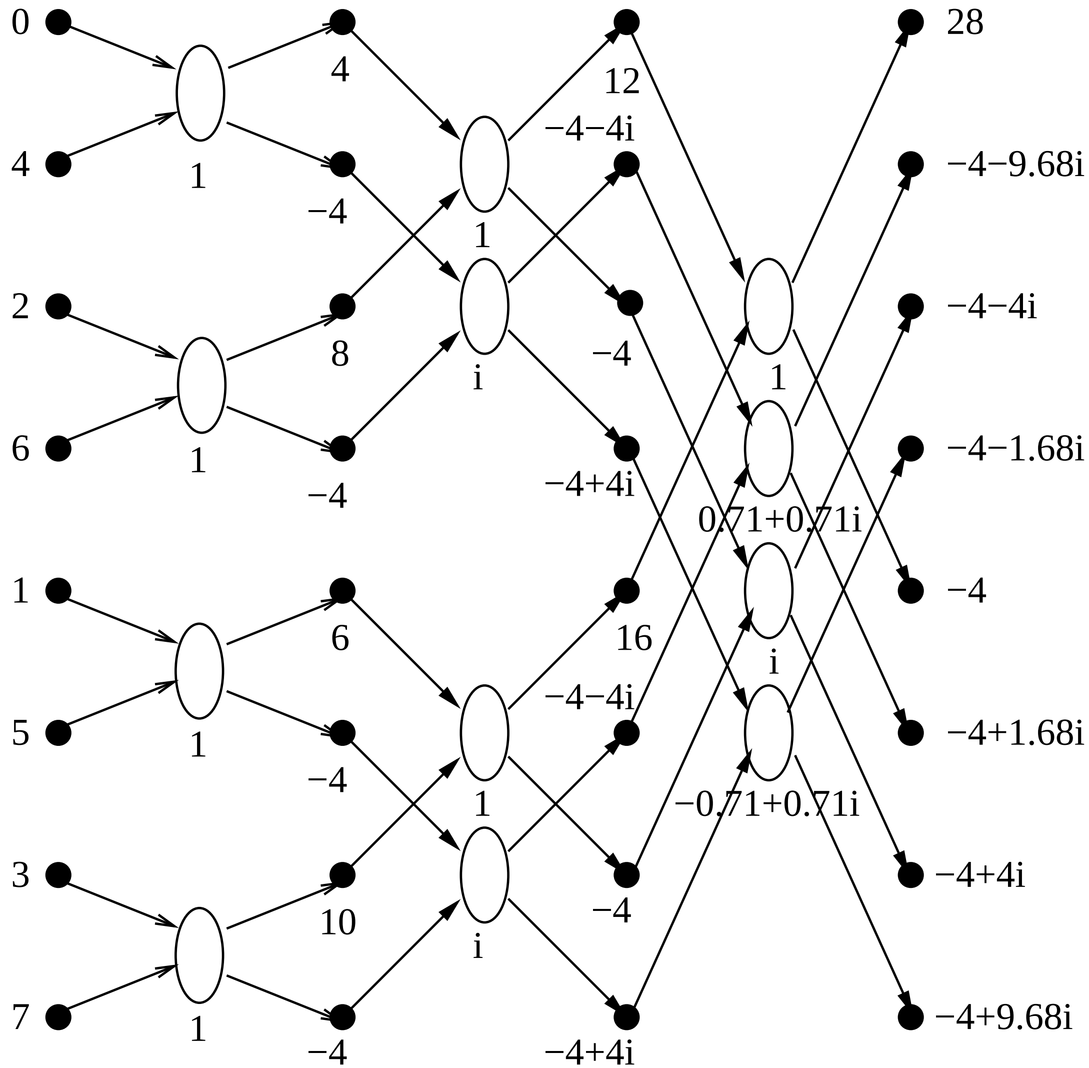
$$\mathbf{y} = F_n \, \mathbf{x}$$

$$y_j = y_j^e + W_n^j y_j^o$$

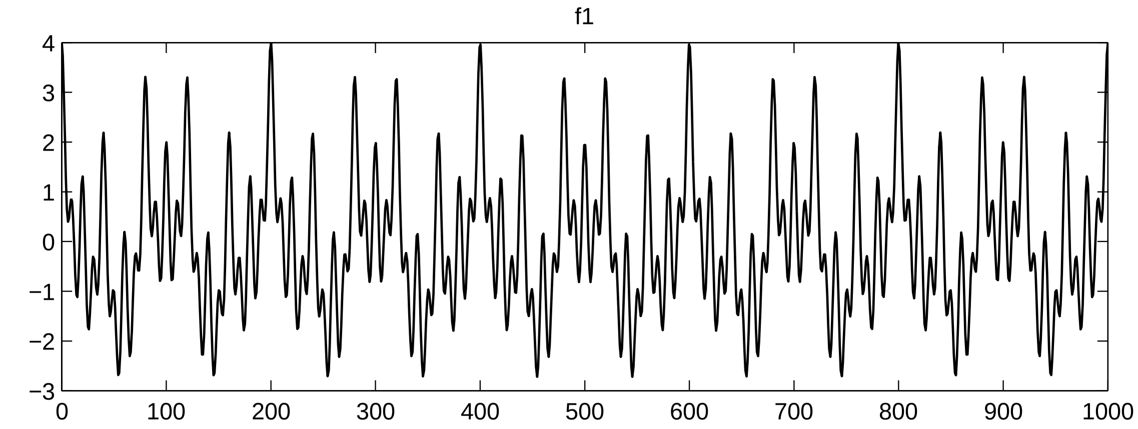
$$y_{m+j} = y_j^e - W_n^j y_j^o$$

$$(j = \overline{0, m-1}, n = 2m)$$

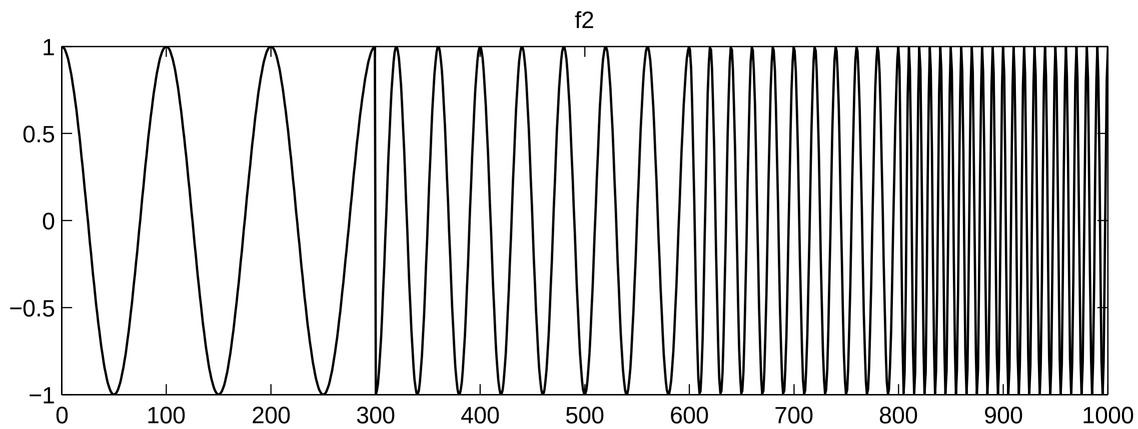
$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{pmatrix} \longrightarrow$$



Stacionaran i nestacionaran signal

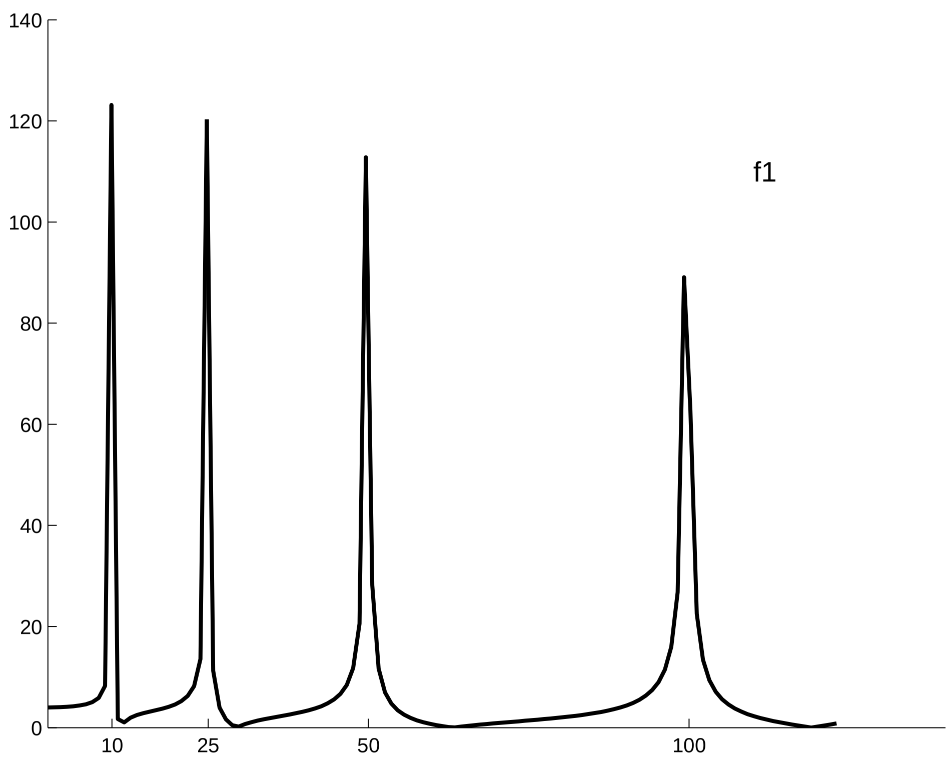


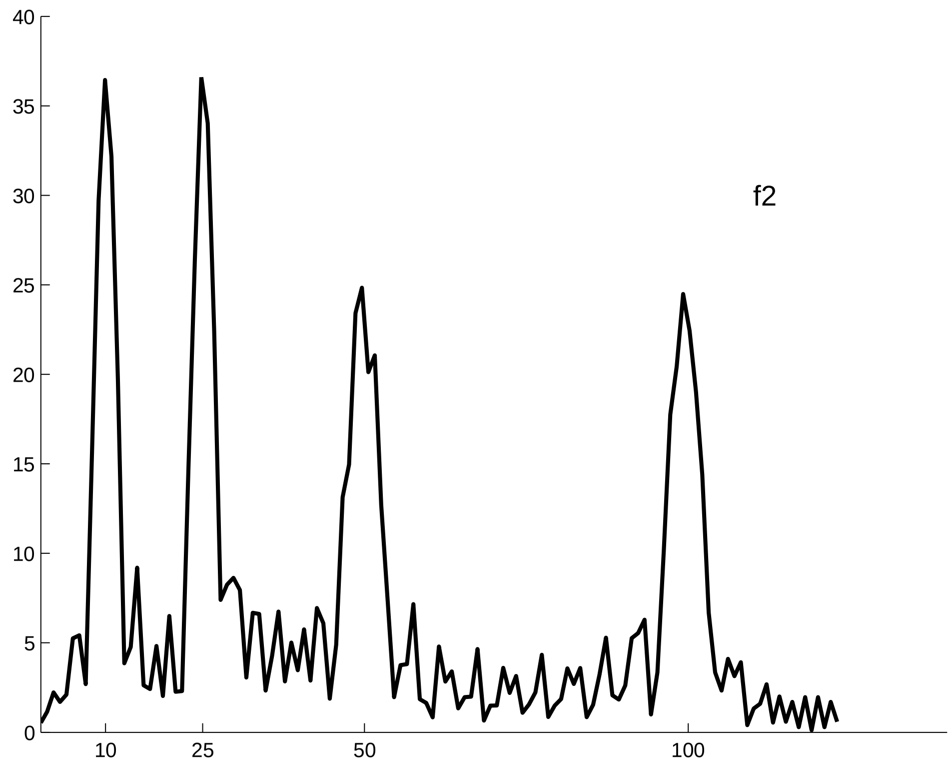
$$\cos(2\pi * 10 * x) + \cos(2\pi * 25 * x) + \cos(2\pi * 50 * x) + \cos(2\pi * 100 * x)$$



$$\begin{cases} \cos(2\pi * 10 * x), & [0, 300] \\ \cos(2\pi * 25 * x), & [300, 600] \\ \cos(2\pi * 50 * x), & [600, 800] \\ \cos(2\pi * 100 * x), & [800, 1000] \end{cases}$$

Fourier-ov spektar



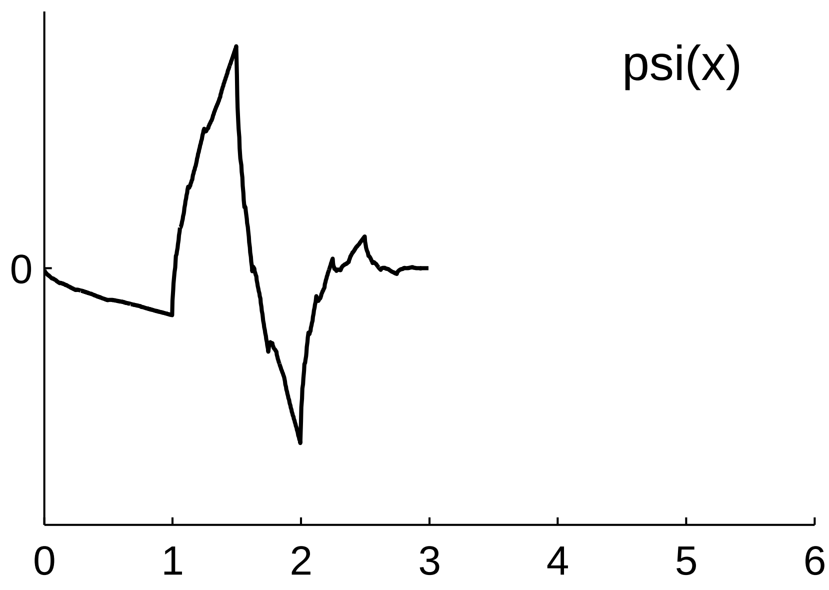


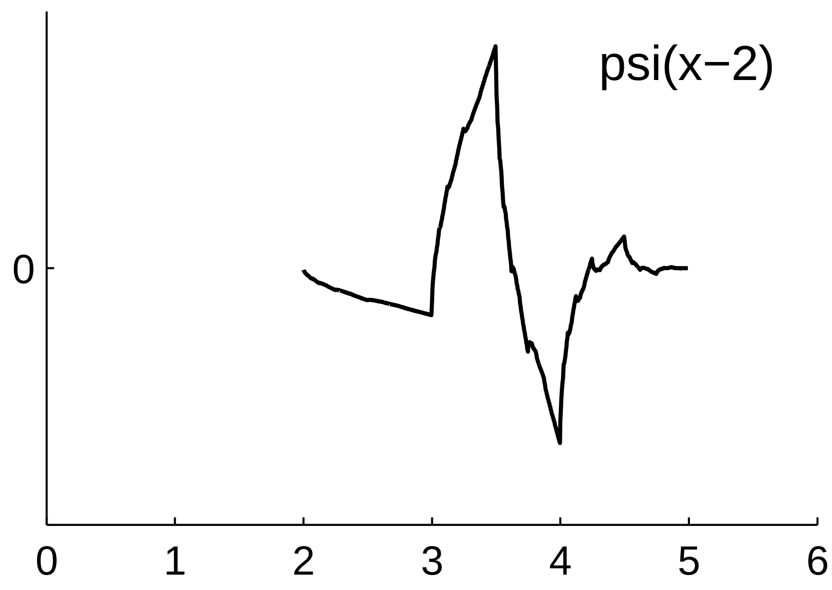
stacionarnog signala, nestacionarnog signala

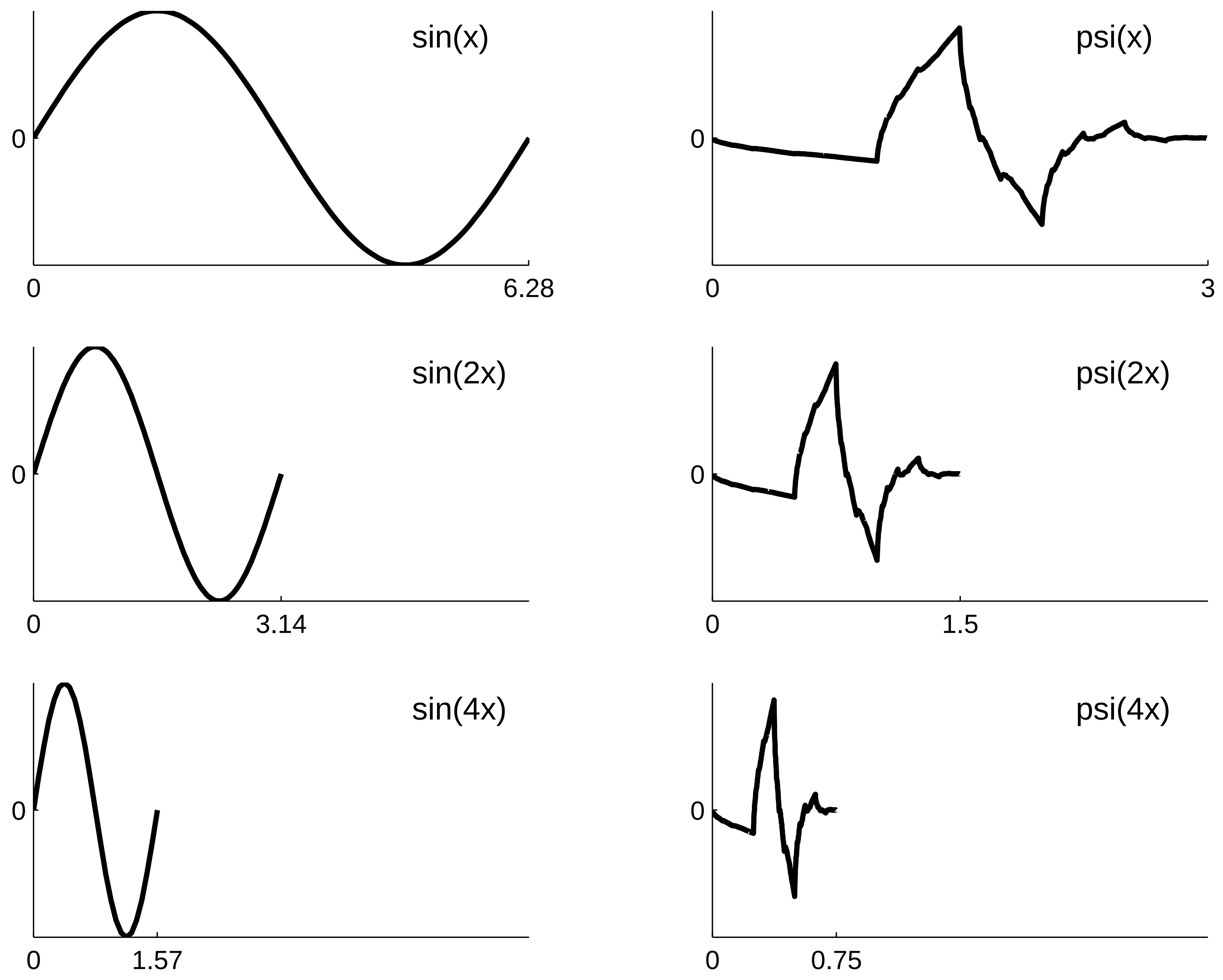
Talasić – oscilatorna funkcija sa kompaktnim nosačem

$$\boxed{\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right)}$$

Translacija talasića (parametar b)



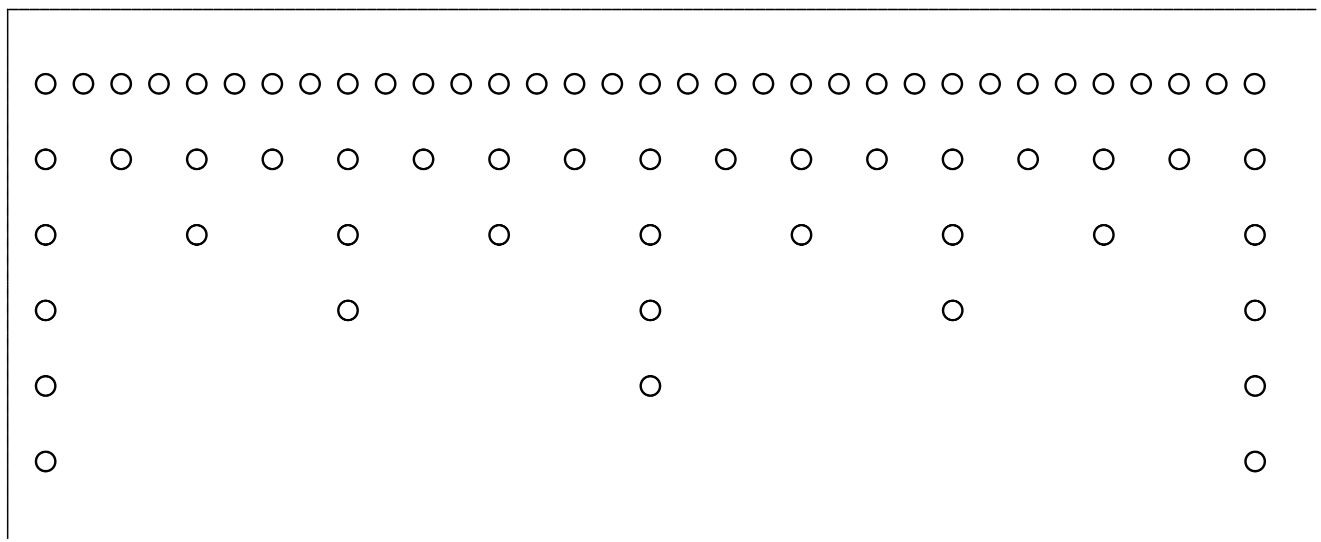




Dilatacija sinusoide i talasića (param. a)

Diskretni talasići $a = 2^j, \quad b = k\,2^j, \qquad k, j \in \mathcal{Z}$

$$\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x - k), \qquad \psi_{jk}(x) \neq 0, \quad x \in [2^j k, 2^j(k+1)].$$



♠ Multirezolucija prostora $\mathcal{L}_2$

(a) $$\cdots \subset \mathcal{V}_2 \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \mathcal{V}_{-2} \subset \dots$$

(b) $$\cap_{j\in Z}\mathcal{V}_j = \{0\}, \qquad \overline{\cup_{j\in Z}\mathcal{V}_j} = \mathcal{L}_2(R)$$

(c) $\forall f \in \mathcal{L}_2(R)$ i $\forall j \in Z,$ $\quad f(x) \in \mathcal{V}_j \quad \Leftrightarrow \quad f(2x) \in \mathcal{V}_{j-1}$

(d) $\forall f \in \mathcal{L}_2(R)$ i $\forall k \in Z,$ $\quad f(x) \in \mathcal{V}_0 \quad \Leftrightarrow \quad f(x-k) \in \mathcal{V}_0$

(e) $\exists \varphi \in \mathcal{V}_0$ tako da je $\{\varphi(x-k)\}_{k\in Z}$ Rieszov bazis u $\mathcal{V}_0$.

$\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x-k), \quad j,k \in Z;$ $\qquad \{\varphi_{j,k}(x)\}_{k\in Z}$ Riesz-ov bazis u $\mathcal{V}_j$

Dilataciona jednačina $$\varphi(x) = \sum_{k=0}^{N-1} c(k)\sqrt{2}\varphi(2x-k), \qquad \int \varphi(x)\,dt = 1$$

Prostor talasića $\mathcal{W}_j$: $\qquad \mathcal{V}_{j-1} = \mathcal{V}_j \oplus \mathcal{W}_j, \quad j \in Z$

Talasić "*majka*" $\psi(x) \in \mathcal{W}_0$ definisan je *jednačinom talasića*

$$\psi(x) = \sum_{k=0}^{N-1} d(k)\sqrt{2}\varphi(2x-k)$$

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x-k) \quad k \in Z; \qquad \{\psi_{j,k}(x)\}_{k\in Z} \text{ bazis u } \mathcal{W}_j$$

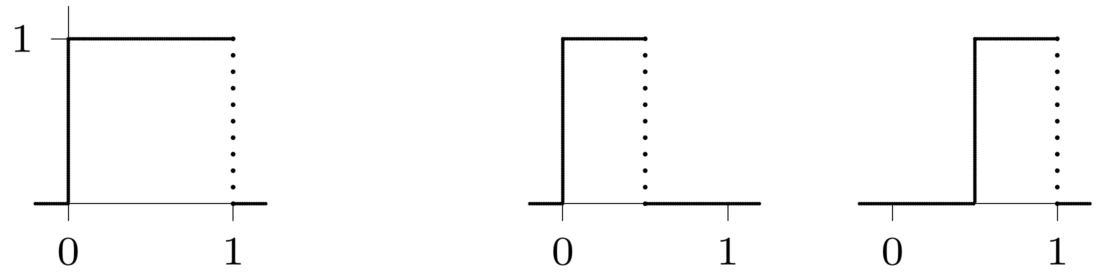
Multirezolucijski razvoj

$$f(x) = \sum_{j\in\mathcal{Z}} \sum_{k\in\mathcal{Z}} b_{j,k}\,\psi_{j,k}(x) = \sum_{k\in\mathcal{Z}} a_{J,k}\,\varphi_{j,k}(x) + \sum_{j=-\infty}^{J} \sum_{k\in\mathcal{Z}} b_{j,k}\,\psi_{j,k}(x)$$

$$\mathcal{V}_J \qquad \oplus \qquad \mathcal{W}_J \oplus \mathcal{W}_{J-1} \oplus \cdots$$

◇

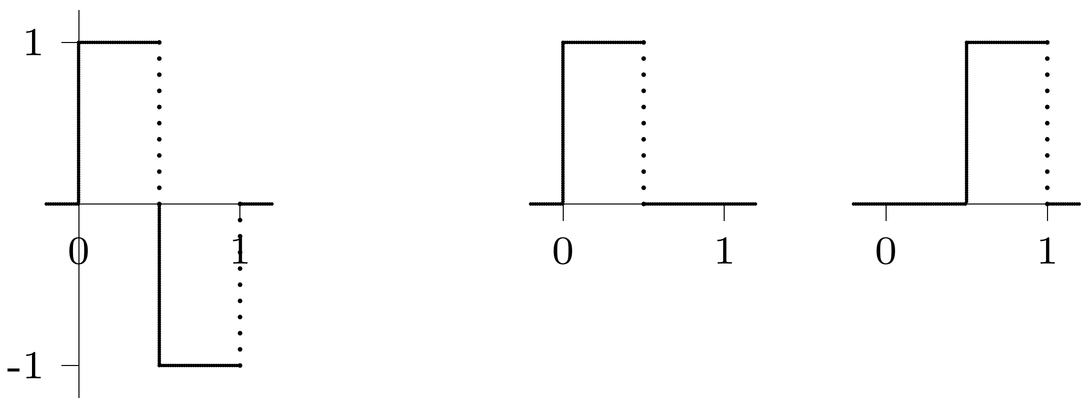
$$\varphi(x) = \varphi(2x) + \varphi(2x-1)$$



Haar-ova četvrtka

$$c(0) = c(1) = \frac{1}{\sqrt{2}}$$

$$\psi(x) = \varphi(2x) - \varphi(2x-1)$$



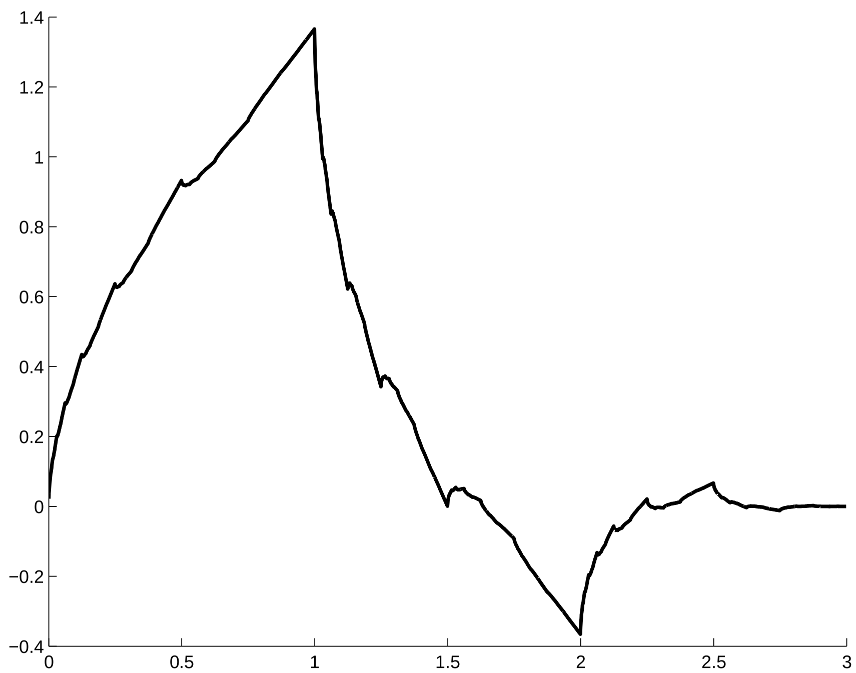
Haar-ov talasić

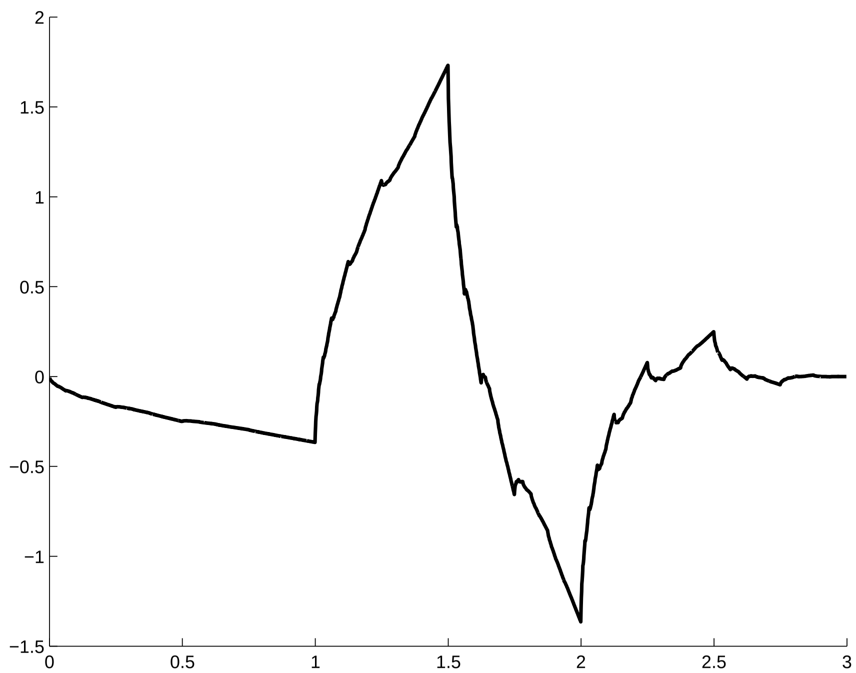
$$d(0) = -d(1) = \frac{1}{\sqrt{2}}$$

http://www.matf.bg.ac.yu/ dradun/talasic1024.html

◇ Daubechies Db2 funkcija skaliranja i talasić (ortogonalni sistem)

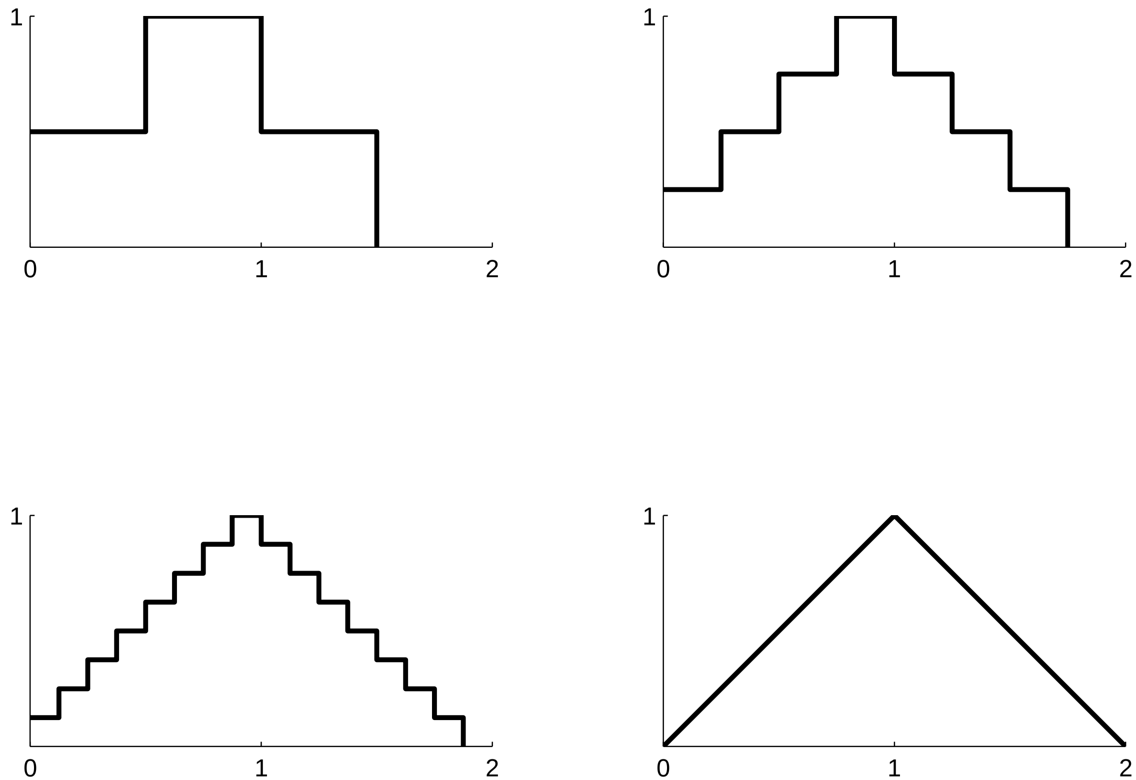
$$d(0) = c(3) = \frac{1-\sqrt{3}}{4\sqrt{2}}, \qquad d(1) = -c(2) = -\frac{3-\sqrt{3}}{4\sqrt{2}},$$
$$d(2) = c(1) = \frac{3+\sqrt{3}}{4\sqrt{2}}, \qquad d(3) = -c(0) = -\frac{1+\sqrt{3}}{4\sqrt{2}}$$





◇ Generisanje linearnog splajna kaskadnim algoritmom: $\varphi^{(0)}(x)$ je četvrtka,

$$\varphi^{(n+1)}(x) = \frac{1}{2}\varphi^{(n)}(2x) + \varphi^{(n)}(2x-1) + \frac{1}{2}\varphi^{(n)}(2x-2), \quad n = 0, 1, \ldots.$$



Piramidalni algoritam – dekompozicija

$$a_{j,k} = \sum_l c(l-2k) a_{j-1,l}, \qquad b_{j,k} = \sum_l d(l-2k) a_{j-1,l}$$

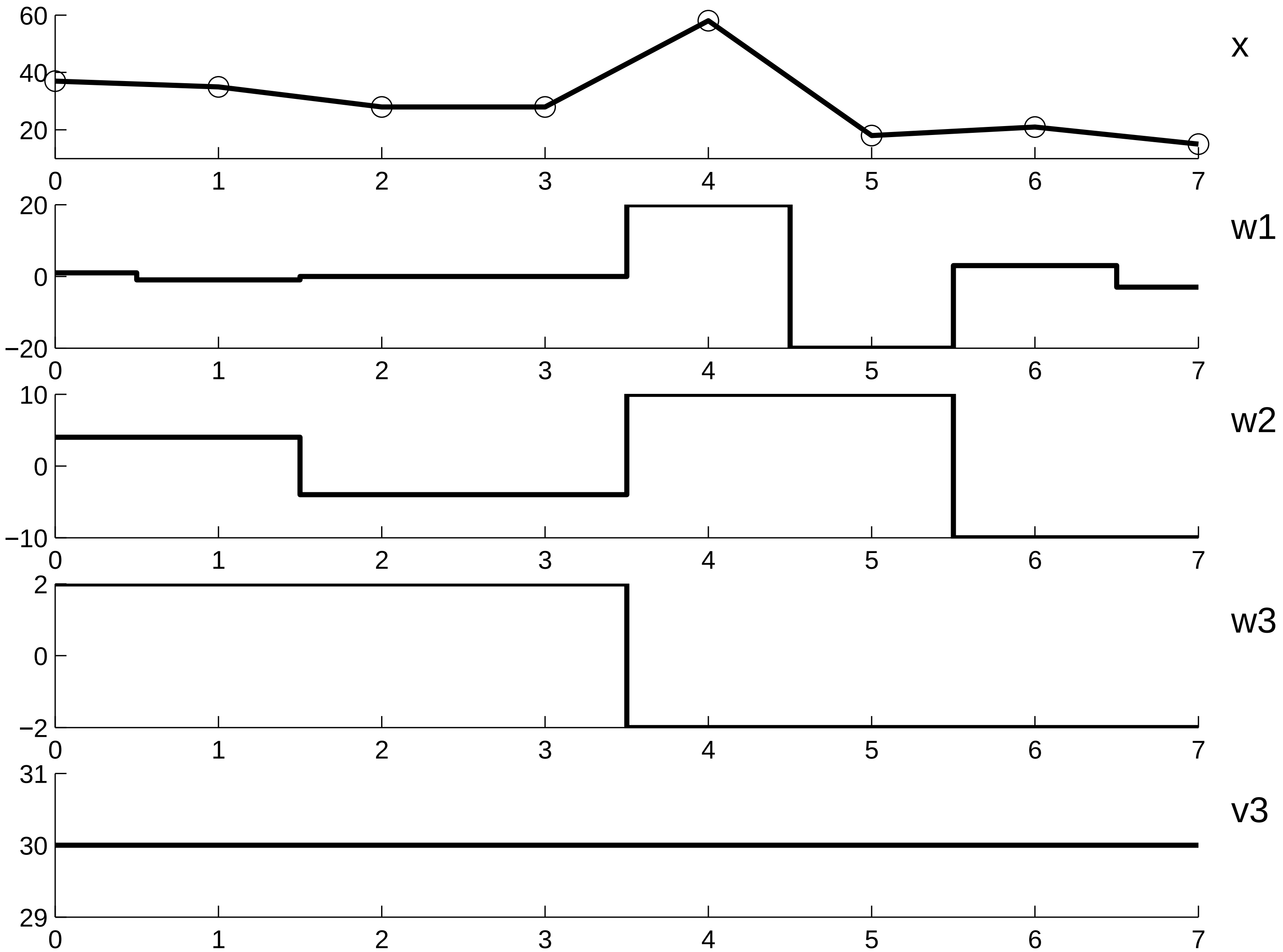
◇

$$c(0) = c(1) = \frac{1}{2}$$

$$d(0) = -d(1) = \frac{1}{2}$$

37	35	28	28	58	18	21	15
36	28	38	18	1	0	20	3
32	28	4	10				
30	2						
30	2	4	10	1	0	20	3

Razlaganje signala piramidalnim algoritmom



Piramidalni algoritam – rekonstrukcija

$$a_{j-1,l} = \sum_k \left(c(l-2k)a_{j,k} + d(l-2k)b_{j,k} \right)$$

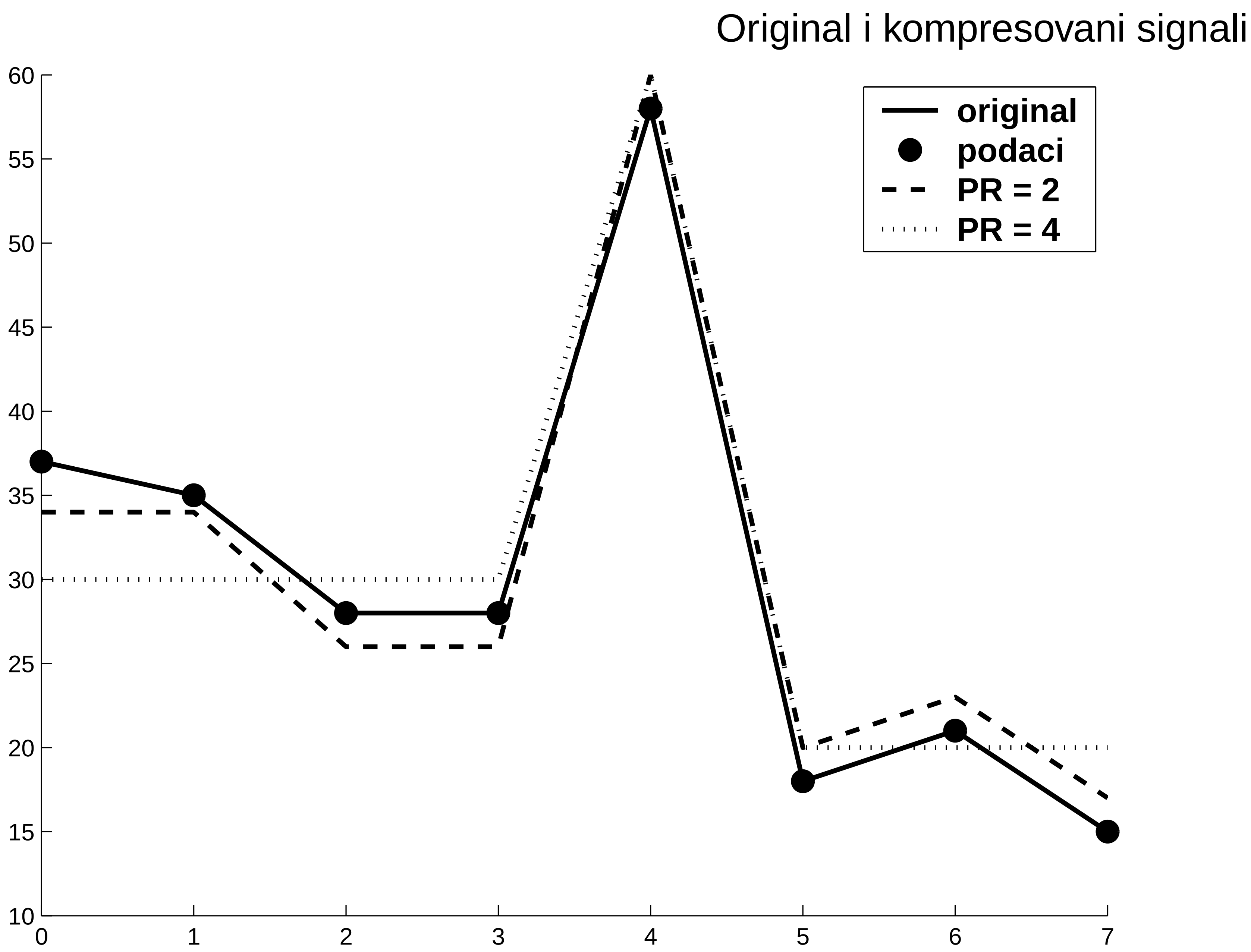
Kompresija* $prag = 2$

30	2	4	10	1	0	20	3
30	0	4	10	0	0	20	3
30	30	4	10	0	0	20	3
34	26	40	20	0	0	20	3
34	34	26	26	60	20	23	17

$prag = 4$

30	2	4	10	1	0	20	3
30	0	0	10	0	0	20	0
30	30	0	10	0	0	20	0
30	30	40	20	0	0	20	0
30	30	30	30	60	20	20	20

*$c(0)=c(1)=d(0)=-d(1)=1$

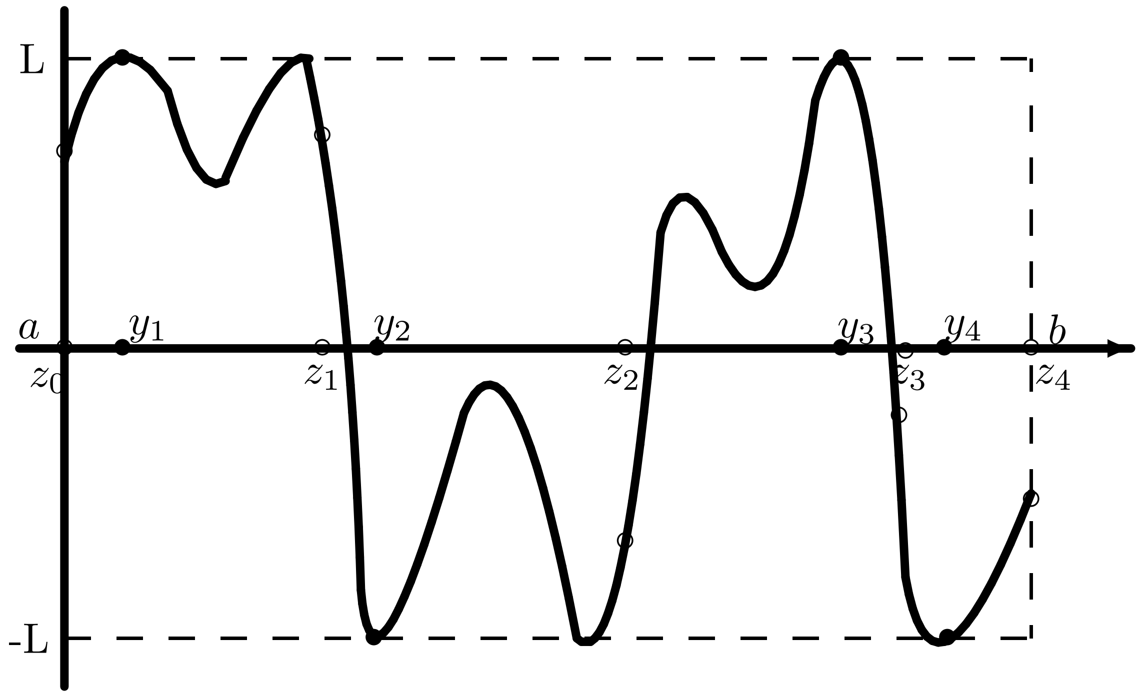

Original i kompresovani signali
original
podaci
PR = 2
PR = 4
60
55
50
45
40
35
30
25
20
15
10
0
1
2
3
4
5
6
7

- *Obrada signala* (analiza, sinteza, kompresija)
  - lociranje i predviđanje zemljotresa,
  - proučavanje udaljenih galaksija,
  - analiza i kompresija medicinskih signala (ECG, EEG),
  - kontrola kvaliteta analizom zvučnog signala,
  - komunikacije (kompresija).
- *Obrada slike*
  - kompresija otisaka prstiju u odnosu 20:1 (JPEG 2000),
  - kompresija slike,
  - kompjuterska grafika (uzastopno renderisanje),
  - kompjuterska vizija (multirezolucijski pristup).
- *Numeričke metode*
  - teorija aproksimacija,
  - multigrid tehnika,
  - modeliranje diferencijalnim jednačinama.

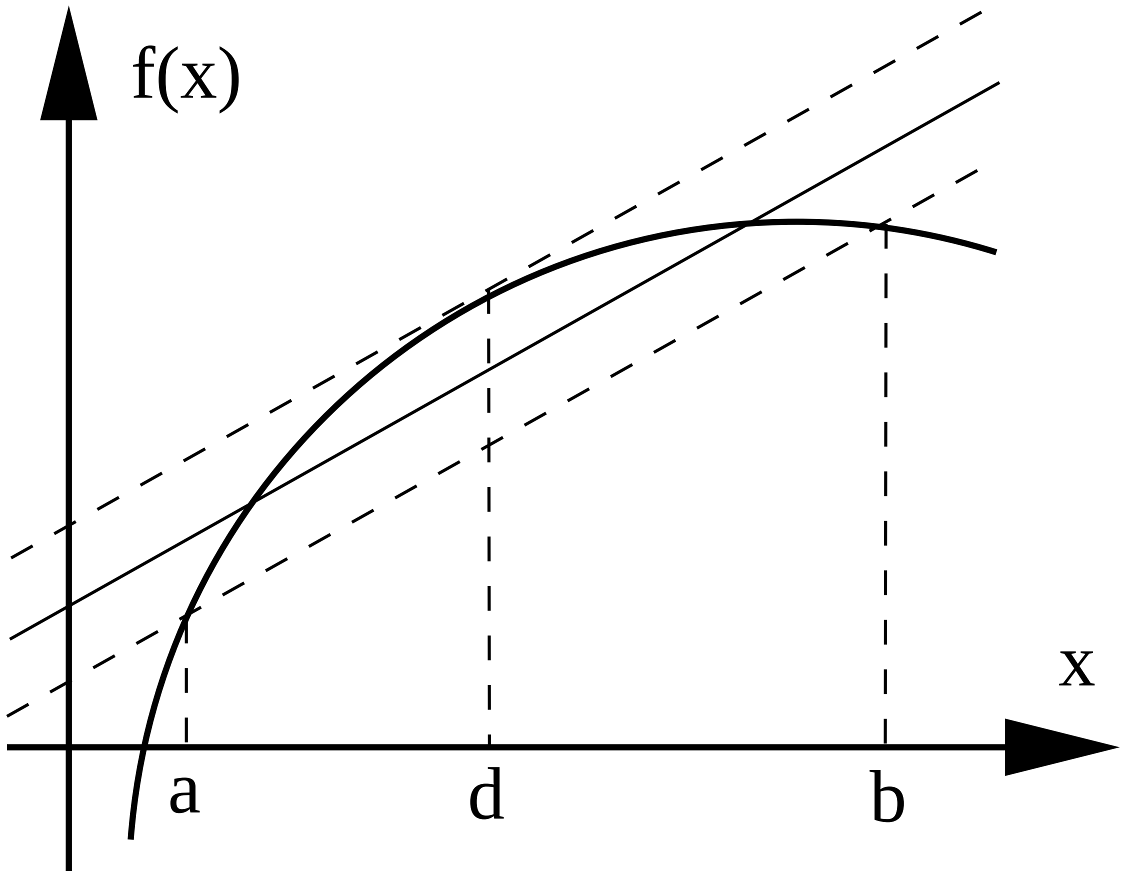
RAVNOMERNA aproksimacija ($\mathcal{C}$)

$$\|f\| = \sup_{x\in[a,b]} |f(x)|, \qquad E_n(f) = \inf_{\mathbf{c}} \left(\sup_{x\in[a,b]} \left| f - \sum_{i=0}^{n} c_i g_i(x) \right| \right)$$

♠ Čebišev: $f(x_i) - Q_0(x_i) = \alpha(-1)^i \|f - Q_0\|,$ $\quad i=\overline{0,n+1}, \quad \alpha=\pm 1$



◇ Aproksimacija konkavne funkcije pravom $\quad Q_0(x) = c_0 + c_1 x$



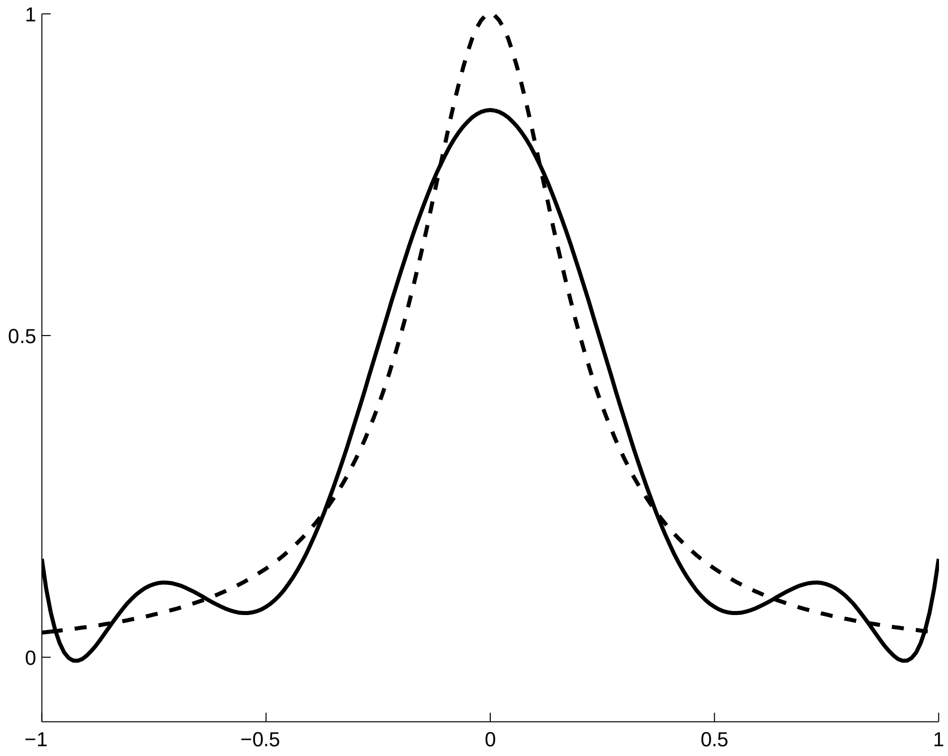
$$f(a) - c_0 - c_1\, a = \alpha L$$

$$f(d) - c_0 - c_1\, d = -\alpha L$$

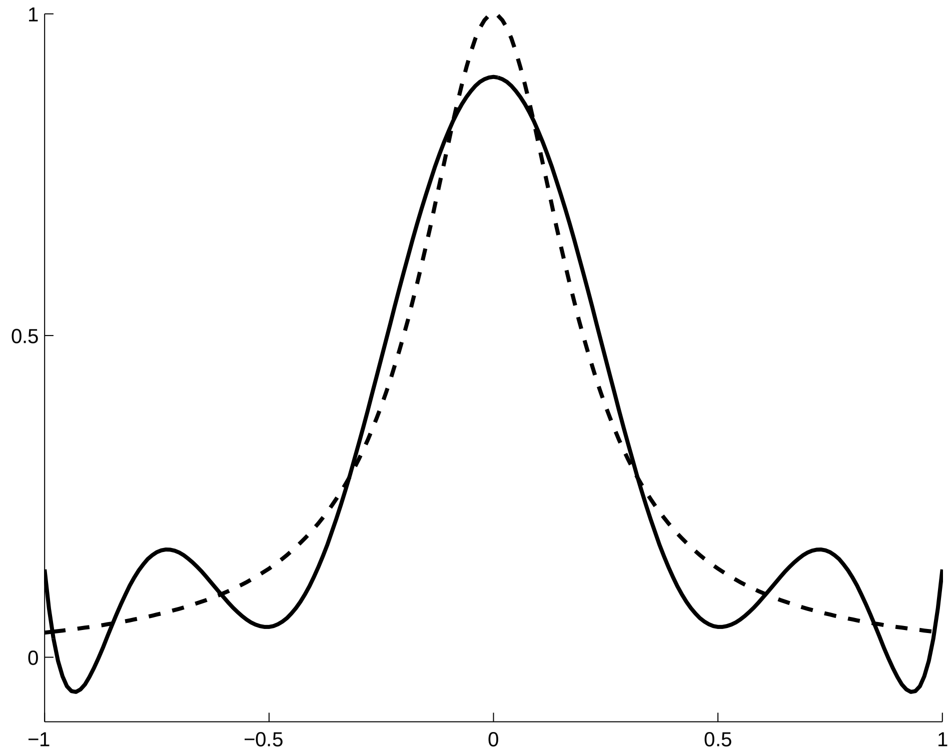
$$f(b) - c_0 - c_1\, b = \alpha L$$

$$f'(d) - Q_0'(d) = 0$$

◇ Aproksimacija funkcije $\frac{1}{1+25x^2}$ polinomom osmog stepena



srednjekvadratna,



ravnomerna

Polinomi Čebiševa $\quad T_n(x) = \cos(n \arccos x), \quad n = 1, \ldots.$

$$T_0(x) = 1, \quad T_1(x) = x, \qquad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$$T_n(x) = \frac{1}{2}\left(x + \sqrt{x^2 - 1}\right)^n + \frac{1}{2}\left(x - \sqrt{x^2 - 1}\right)^n$$

Ortogonalnost $\quad (T_n\,,\,T_m) \equiv \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1 - x^2}}\,dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$

♠ Najmanje odstupanje od 0

$$\max_{x\in[-1,1]} |P_n(x)| \geq \max_{x\in[-1,1]} |2^{1-n} T_n(x)| = 2^{1-n}$$