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THE ZONAL SATELLITE PROBLEM. III. SYMMETRIES

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SUMMARY: The two-body problem associated with a force field described by a potential of the form $U = \sum_{k=1}^{n} a_k/r^k$ (r = distance between particles, a_k = real parameters) is resumed from the only standpoint of symmetries. Such symmetries, expressed in Hamiltonian coordinates, or in standard polar coordinates, are recovered for McGehee-type coordinates of both collision-blow-up and infinity-blow-up kind. They form diffeomorphic commutative groups endowed with a Boolean structure. Expressed in Levi-Civita's coordinates, the problem exhibits a larger group of symmetries, also commutative and presenting a Boolean structure.

1. INTRODUCTION

The name of *zonal satellite problem* was assigned by Mioc and Stavinschi (1998a; hereafter Paper I) to a special class of two-body problems, the one associated with a potential of the form $U = \sum_{k=1}^{n} a_k/r^k$, where r is the distance between particles, while a_k are real parameters. This denomination is due to the fact that the best known potential of this type is that represented by the zonal part of a planetary gravitational potential (see also Cid *et al.* 1983).

This class of problems of particle nonlinear dynamics generalizes a lot of classical models, as those of Kepler, Manev, Schwarzschild, Fock, Reissner-Nordström, Coulomb, Van der Vaals, etc. It also has implications in astrophysics, mechanics, celestial mechanics and dynamical astronomy, space dynamics, even atomic physics. To have an idea about the degree of generality of the zonal satellite problem, see Paper I and the references therein.

A first qualitative insight into the zonal satellite problem was offered in Paper I. A chain of McGehee-type transformations of the second kind (McGehee 1974) was used to blow up the collision singularity and to obtain regular equations of motion and first integrals. Then the flow on the collision manifold and in its neighbourhood was described. A second step was performed also by Mioc and Stavinschi (1998b; hereafter Paper II), who tackled the infinity manifold, and depicted the flow on it and in its neighbourhood.

In this paper we resume the zonal satellite problem from a single standpoint: symmetries. In Section 2 we invoke the basic equations of the problem in Hamiltonian coordinates. The equations of motion present nice symmetries that form a commutative group endowed with a Boolean structure.

In Section 3 we refer to the McGehee transformations that make the singularity blow up and replace it by a manifold pasted on the phase space. Dwelling on the vector field that describes the problem in standard polar coordinates and velocity components, and in physical time, we point out the symmetries it holds. These symmetries also form a commutative group endowed with a Boolean structure.

Section 4 tackles the equations of motion expressed in McGehee-type collision-blow-up coordinates. These equations also exhibit symmetries, which form a group endowed with the same features as those described in Sections 2 and 3.

In Section 5 we recall the McGehee-type infinity-blow-up coordinates, intended to extend the problem to the limiting case of escape/capture. Expressed in these coordinates, the vector field has symmetries that form another group, presenting the same characteristics as those pointed out in Sections 2, 3, and 4.

Just to make a comparison with other regularizing transformations, in Section 6 we transpose the initial equations into Levi-Civita's (1903) coordinates. The corresponding vector field also exhibits symmetries, which form a doubly larger group than the previous ones. The commutativity and the Boolean structure are kept.

These results are of much help in better understanding the local flows of the problem in the neighbourhood of both collision/ejection and escape/capture, as well as some features of the global flow.

2. SYMMETRIES IN HAMILTONIAN COORDINATES

Reducing the two-body problem to a central-force problem, Mioc and Stavinschi (Paper I) wrote the corresponding equations of motion in canonical formalism:

$$\begin{aligned}\dot{\mathbf{q}} &= \partial H(\mathbf{q},\mathbf{p})/\partial \mathbf{p},\\ \dot{\mathbf{p}} &= -\partial H(\mathbf{q},\mathbf{p})/\partial \mathbf{q},\end{aligned} \tag{1}$$

with the Hamiltonian

$$H(\mathbf{q},\mathbf{p}) = |\mathbf{p}|^2/2 - \sum_{k=1}^{n} a_k/|\mathbf{q}|^k. \tag{2}$$

Recall that $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2 \backslash \{(0,0)\}$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$ are the position (configuration) vector and the momentum vector of one body (particle) with respect to the other (centre), respectively.

Remark 1. The equations of motion admit the first integrals of energy and angular momentum (Paper I; Mioc and Stavinschi 2000). However, since they play no role in the present treatment, we shall leave them aside.

Let us write the vector field (1) explicitly. By (2), it reads

$$\begin{aligned}\dot{q}_1 &= p_1,\\ \dot{q}_2 &= p_2,\\ \dot{p}_1 &= -\sum_{k=1}^{n} k a_k (q_1^2+q_2^2)^{-k/2-1} q_1,\\ \dot{p}_2 &= -\sum_{k=1}^{n} k a_k (q_1^2+q_2^2)^{-k/2-1} q_2.\end{aligned} \tag{3}$$

Proposition 1. *The vector field (3) holds seven symmetries, as follows:*

$$\begin{aligned}S_1(q_1,q_2,p_1,p_2,t) &= (q_1,q_2,-p_1,-p_2,-t),\\ S_2(q_1,q_2,p_1,p_2,t) &= (q_1,-q_2,p_1,-p_2,t),\\ S_3(q_1,q_2,p_1,p_2,t) &= (-q_1,q_2,-p_1,p_2,t),\\ S_4(q_1,q_2,p_1,p_2,t) &= (q_1,-q_2,-p_1,p_2,-t),\\ S_5(q_1,q_2,p_1,p_2,t) &= (-q_1,q_2,p_1,-p_2,-t),\\ S_6(q_1,q_2,p_1,p_2,t) &= (-q_1,-q_2,-p_1,-p_2,t),\\ S_7(q_1,q_2,p_1,p_2,t) &= (-q_1,-q_2,p_1,p_2,-t).\end{aligned} \tag{4}$$

Proof. Examining equations (3), it is easy to show that they are invariant to the transformations described by (4). □

Proposition 2. *Out of the seven symmetries (4), only three are independent.*

Proof. Consider, for instance that the symmetries S_1, S_2, S_3 are mutually independent. It is easy to show that

$$\begin{aligned}S_4 &= S_1 \circ S_2,\\ S_5 &= S_1 \circ S_3,\\ S_6 &= S_2 \circ S_3,\\ S_7 &= S_1 \circ S_2 \circ S_3.\end{aligned} \tag{5}$$

Choosing arbitrarily three different symmetries in $\{S_i \mid i = \overline{1,7}\}$, and considering them as independent of each other, an analogy to the above structure is established. The proposition is proved. □

Theorem 1. *The set $G = \{I\} \cup \{S_i \mid i = \overline{1,7}\}$ (where I denotes the identity), endowed with the composition law "$\circ$" forms a symmetric, commutative group.*

Proof. By virtue of Propositions 1 and 2, it is easy to construct the composition table

$$\begin{bmatrix} \circ & I & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ I & I & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ S_1 & S_1 & I & S_4 & S_5 & S_2 & S_3 & S_7 & S_6 \\ S_2 & S_2 & S_4 & I & S_6 & S_1 & S_7 & S_3 & S_5 \\ S_3 & S_3 & S_5 & S_6 & I & S_7 & S_1 & S_2 & S_4 \\ S_4 & S_4 & S_2 & S_1 & S_7 & I & S_6 & S_5 & S_3 \\ S_5 & S_5 & S_3 & S_7 & S_1 & S_6 & I & S_4 & S_2 \\ S_6 & S_6 & S_7 & S_3 & S_2 & S_5 & S_4 & I & S_1 \\ S_7 & S_7 & S_6 & S_5 & S_4 & S_3 & S_2 & S_1 & I \end{bmatrix}$$

that proves the theorem. □

Corollary 1. *The group G is endowed with a Boolean structure.*

Proof. Examining the composition table above, it is clear that every element is its own inverse with respect to the composition law "$\circ$".The result is proved. □

3. SYMMETRIES IN STANDARD POLAR COORDINATES

To remove the isolated singularity at the origin $\mathbf{q} = (0, 0)$, which corresponds to a collision particle-centre (e.g. Mioc and Stavinschi 2001, 2002), in Paper I the following chain of McGehee-type transformations of the second kind (McGehee 1974) was used:

$$\begin{aligned} r &= |\mathbf{q}|, \\ \theta &= \arctan(q_2/q_1), \\ \xi &= \dot{r} = (q_1p_1 + q_2p_2)/|\mathbf{q}|, \\ \eta &= r\dot{\theta} = (q_1p_2 - q_2p_1)/|\mathbf{q}|, \end{aligned} \tag{6}$$

which introduces the standard polar coordinates and the polar components of the velocity;

$$\begin{aligned} x &= r^{n/2}\xi, \\ y &= r^{n/2}\eta. \end{aligned} \tag{7}$$

which scale down the components of the velocity;

$$ds = r^{-n/2-1}dt. \tag{8}$$

which rescales the time. Recall that all these transformations are real analytic diffeomorphisms.

Let us dwell, for the moment, on the vector field that results after the transformations (6). Its expression is (Paper I):

$$\begin{aligned} \dot{r} &= \xi, \\ \dot{\theta} &= \eta/r, \\ \dot{\xi} &= \eta^2/r - \sum_{k=1}^{n} ka_k/r^{k+1}, \\ \dot{\eta} &= -\xi\eta/r, \end{aligned} \tag{9}$$

This nonregular vector field describes the motion in standard polar coordinates. Supposing that $0 < r < +\infty$, we may transpose the remarkable property of this model pointed out in Section 2 under the form of

Proposition 3. *The vector field (9) holds of seven symmetries, as follows:*

$$\begin{aligned} S_1^{pol}(r,\theta,\xi,\eta,t) &= (r,\theta,-\xi,-\eta,-t), \\ S_2^{pol}(r,\theta,\xi,\eta,t) &= (r,-\theta,\xi,-\eta,t), \\ S_3^{pol}(r,\theta,\xi,\eta,t) &= (r,\pi-\theta,\xi,-\eta,t), \\ S_4^{pol}(r,\theta,\xi,\eta,t) &= (r,-\theta,-\xi,\eta,-t), \\ S_5^{pol}(r,\theta,\xi,\eta,t) &= (r,\pi-\theta,-\xi,\eta,-t), \\ S_6^{pol}(r,\theta,\xi,\eta,t) &= (r,\pi+\theta,\xi,\eta,t), \\ S_7^{pol}(r,\theta,\xi,\eta,t) &= (r,\pi+\theta,-\xi,-\eta,-t). \end{aligned} \tag{10}$$

Proof. Examining equations (9), it is easy to show that they are invariant to the transformations described by (10). □

Let us see what these symmetries mean. Considering separately the variables, $(t, -t)$ corresponds to motion in the future/past; $(\xi, -\xi)$ means outwards/inwards motion; $(\eta, -\eta)$ means clockwise/counterclockwise motion; finally, $(\theta, -\theta)$, $(\theta, \pi - \theta)$, $(\theta, \pi + \theta)$ correspond to positions shifted mutually by 2θ, $\pi - 2\theta$, and π, respectively. As to their combinations into symmetries, S_1^{pol} corresponds to the reversibility of the flow: for every solution, there is another solution that has the same position coordinates and opposite velocities, all in reversed time. S_2^{pol} implies that, for every solution, there is another solution with opposite θ and η coordinates, and so forth.

Proposition 4. *The symmetries (10) are diffeomorphically the same as the symmetries (4).*

Proof. Defining

$$S_i^{pol}(r,\theta,\xi,\eta,t) = S_i(q_1,q_2,p_1,p_2,t), \quad i = \overline{1,7}, \tag{11}$$

and taking into account the real analytic diffeomorphisms (6), the statement is obvious. □

Proposition 4 enables us to state (without proof) the following results:

Proposition 5. *Out of the symmetries (10), only three are independent.*

Theorem 2. *The set $G^{pol} = \{I\} \cup \{S_i^{pol} \mid i = \overline{1,7}\}$, endowed with the composition law "$\circ$", forms a symmetric, commutative group.*

Corollary 2. *The group G^{pol} is endowed with a Boolean structure.*

4. SYMMETRIES IN COLLISION-BLOW--UP COORDINATES

Let us return to the nonregular vector field (9). Under the next two steps in the sequence of McGehee-type transformations, (7) and (8), it becomes (Paper I):

$$\begin{aligned} r' &= rx, \\ \theta' &= y, \\ x' &= nx^2/2 + y^2 - \sum_{k=1}^{n} k a_k r^{n-k}, \\ y' &= (n/2-1)xy, \end{aligned} \tag{12}$$

where $' = d/ds$ and we kept, by abuse, the same notation for the new functions of the timelike variable s.

Recall that, in this way, the collision singularity at $q = (0,0)$ or $r = 0$ was blown up and replaced by a manifold pasted on the phase space (Paper I). The phase space extends smoothly to this manifold.

Perusing the regular equations of motion (12), we can state

Theorem 3. *The vector field (12), corresponding to the phase space extended to the collision boundary, holds the same symmetries as the vector field (9).*

Proof. Taking into account the symmetries (10), let us define

$$\tilde{S}_i(r,\theta,x,y,s) = S_i^{pol}(r,\theta,\xi,\eta,t), \quad i = \overline{1,7}, \tag{13}$$

Examining equations (12), and taking into account (10), it is easy to verify their invariance to the transformations (13). □

Given the definition of the symmetries $\tilde{S}_i$, $i = \overline{1,7}$, the following result is immediate:

Proposition 6. *Out of the seven symmetries $\tilde{S}_i$, only three are independent.*

To end this section, we state its main result, which was to be expected:

Theorem 4. *The set $G_0 = \{I\} \cup \{\tilde{S}_i \mid i = \overline{1,7}\}$, endowed with the same composition law "$\circ$ " as the groups G and G^{pol}, forms a symmetric, commutative group.*

Proof. The analogy to the composition table used in the proof of Theorem 1 is easy to establish and check. The theorem is proved. □

Corollary 3. *The group G_0 is endowed with a Boolean structure.*

Proof. As in the case of Corollary 1, it suffices to examine the composition table corresponding to G_0; every element proves to be its own inverse. □

5. SYMMETRIES IN INFINITY-BLOW-UP COORDINATES

To tackle a somewhat opposite situation, in Paper II was considered the escape case ($r \to \infty$). To make the infinity turn to a singularity, one resorted to the McGehee-type transformation of the first kind (McGehee 1973):

$$\rho = 1/r. \tag{14}$$

To remove the singularity at $\rho = 0$, the following McGehee-type transformation of the second kind was used (McGehee 1974):

$$\begin{aligned} u &= \rho^{n/2} x, \\ v &= \rho^{n/2} y; \end{aligned} \tag{15}$$

$$d\tau = \rho^{-n/2} ds. \tag{16}$$

Under these transformations, which are all real analytic diffeomorphisms, too, the vector field (12) becomes

$$\begin{aligned} d\rho/d\tau &= -\rho u, \\ d\theta/d\tau &= v, \\ du/d\tau &= v^2 - \sum_{k=1}^{n} k a_k \rho^k, \\ dv/d\tau &= -uv, \end{aligned} \tag{17}$$

where we abused again the notation for the new functions of the timelike variable τ.

Remark 2. By (14)−(16), we have blown up the singularity at $\rho = 0$, and replaced it by a manifold pasted on the phase space (Paper II). The phase space extends smoothly to the infinity boundary.

Remark 3. Equations (12) and (17) describe the same problem as (3) or (9), but in different time scales. Each time scale difffers in turn from the "physical" time t (compare (8) and (16)).

As in the case of (9) and (12), the equations of motion (17) possess of special characteristics. We can state

Theorem 5. *The vector field (17), corresponding to the phase space extended to the infinity boundary, has the same symmetries as the vector fields (9) and (12).*

Proof. Let us define

$$\widehat{S}_i(\rho,\theta,u,v,\tau) = S_i(r,\theta,\xi,\eta,t), \quad i = \overline{1,7}, \tag{18}$$

or, equivalently,

$$\widehat{S}_i(\rho,\theta,u,v,\tau) = \tilde{S}_i(r,\theta,x,y,s), \quad i=\overline{1,7}, \tag{19}$$

Examining equations (17), their invariance to the transformations described by $\widehat{S}_i$, $i=\overline{1,7}$, can be easily checked. □

Given the equivalent definitions of the symmetries $\widehat{S}_i$, and constructing the composition table analogous to those corresponding to S_i, S_i^{pol}, and $\tilde{S}_i$, the following results are immediate:

Proposition 7. *Out of the seven symmetries $\widehat{S}_i$, only three are independent.*

Theorem 6. *The set $G_\infty = \{I\} \cup \{\widehat{S}_i \mid i=\overline{1,7}\}$, endowed with the same composition law as the groups G, G^{pol}, and G_0, forms a symmetric, commutative group.*

Corollary 4. *The group G_∞ is endowed with a Boolean structure.*

6. SYMMETRIES IN LEVI-CIVITA COORDINATES

In this study, to avoid singularities, we resorted to McGehee-type transformations. But there is a lot of equations of motion-regularizing transformations we could use. In this section, just to compare the results as regards symmetries, we shall apply Levi-Civita's (1903) transformations − based on Euler's (1767) regularization − to the motion equations (9) (see also Aarseth and Zare 1974; Zare 1974; Şelaru 1997a,b).

Let us apply the first step of Levi-Civita's transformations

$$\begin{aligned} r &= z^2, \\ \dot{r} &= w/z, \\ \dot{\theta} &= \varphi \end{aligned} \tag{20}$$

to the vector field (9). This becomes

$$\begin{aligned} \dot{z} &= w/(2z^2), \\ \dot{\theta} &= \varphi, \\ \dot{w} &= w^2/(2z^3) + z^3\varphi^2 - \sum_{k=1}^{n} ka_k/z^{2k+1}, \\ \dot{\varphi} &= -2w\varphi/z^3. \end{aligned} \tag{21}$$

For the second step, the change of the dynamical variable

$$d\sigma = z^{-2n-1}dt \tag{22}$$

makes the vector field (21) turn to

$$\begin{aligned} dz/d\sigma &= wz^{2n-1}/2, \\ d\theta/d\sigma &= \varphi z^{2n+1}, \\ dw/d\sigma &= w^2 z^{2n-2}/2 + \varphi^2 z^{2n+4} - \sum_{k=1}^{n} ka_k/z^{2n-2k}, \\ d\varphi/d\sigma &= -2w\varphi z^{2n-2}, \end{aligned} \tag{23}$$

where we maintained, by abuse, the same notation for the new functions of the timelike variable σ.

Remark 4. Keeping in view the expression of the potential (Paper I), the case $n=1$ represents a limit, physically concretized by the Newtonian-type potential, characterized by an inverse-square interaction law. (If $a_1>0$, the interaction is attractive, generalizing Newton's law. If $a_1<0$, the interaction is repulsive, as in the radiative case. If $a_1=0$, we are in the degenerate case of the force-free field, mainly created by a balance of opposite forces.) In this case ($n=1$), equations (23) reduce to

$$\begin{aligned} dz/d\sigma &= wz/2, \\ d\theta/d\sigma &= \varphi z^3, \\ dw/d\sigma &= w^2/2 + \varphi^2 z^6 - a_1, \\ d\varphi/d\sigma &= -2w\varphi. \end{aligned} \tag{24}$$

Equations (23) also holds remarkable properties as regards symmetries. Examining them, we can state

Proposition 8. *The vector field (23) has fifteen symmetries, as follows*

$$\begin{aligned}
\bar{S}_1(z,\theta,w,\varphi,\sigma) &= (z,\theta,-w,-\varphi,-\sigma); \\
\bar{S}_2(z,\theta,w,\varphi,\sigma) &= (z,-\theta,w,-\varphi,\sigma); \\
\bar{S}_3(z,\theta,w,\varphi,\sigma) &= (z,\pi-\theta,w,-\varphi,\sigma); \\
\bar{S}_4(z,\theta,w,\varphi,\sigma) &= (-z,\theta,-w,\varphi,-\sigma); \\
\bar{S}_5(z,\theta,w,\varphi,\sigma) &= (z,\theta,-w,\varphi,-\sigma); \\
\bar{S}_6(z,\theta,w,\varphi,\sigma) &= (z,\pi-\theta,-w,\varphi,-\sigma); \\
\bar{S}_7(z,\theta,w,\varphi,\sigma) &= (-z,\theta,w,-\varphi,\sigma); \\
\bar{S}_8(z,\theta,w,\varphi,\sigma) &= (z,\pi+\theta,w,\varphi,\sigma); \\
\bar{S}_9(z,\theta,w,\varphi,\sigma) &= (-z,-\theta,-w,-\varphi,-\sigma); \\
\bar{S}_{10}(z,\theta,w,\varphi,\sigma) &= (-z,\pi-\theta,-w,-\varphi,-\sigma); \\
\bar{S}_{11}(z,\theta,w,\varphi,\sigma) &= (z,\pi+\theta,-w,-\varphi,-\sigma); \\
\bar{S}_{12}(z,\theta,w,\varphi,\sigma) &= (-z,-\theta,w,\varphi,\sigma); \\
\bar{S}_{13}(z,\theta,w,\varphi,\sigma) &= (-z,\pi-\theta,w,\varphi,\sigma); \\
\bar{S}_{14}(z,\theta,w,\varphi,\sigma) &= (-z,\pi+\theta,-w,\varphi,-\sigma); \\
\bar{S}_{15}(z,\theta,w,\varphi,\sigma) &= (-z,\pi+\theta,w,-\varphi,\sigma).
\end{aligned} \tag{25}$$

Proof. The invariance of equations (23) to these transformations can be immediately verified.

Remark 5. Obviously, the Kepler-type vector field (24) (associated with the Newtonian potential) holds the same symmetries.

Proposition 9. *Out of the fifteen symmetries, only four are independent.*

Proof. Consider, for instance, that the four mutually independent symmetries are $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$, $\bar{S}_4$. It is easy to show that

$$\begin{aligned}
\bar{S}_5 &= \bar{S}_1 \circ \bar{S}_2,\\
\bar{S}_6 &= \bar{S}_1 \circ \bar{S}_3,\\
\bar{S}_7 &= \bar{S}_1 \circ \bar{S}_4,\\
\bar{S}_8 &= \bar{S}_2 \circ \bar{S}_3,\\
\bar{S}_9 &= \bar{S}_2 \circ \bar{S}_4,\\
\bar{S}_{10} &= \bar{S}_3 \circ \bar{S}_4,\\
\bar{S}_{11} &= \bar{S}_1 \circ \bar{S}_2 \circ \bar{S}_3,\\
\bar{S}_{12} &= \bar{S}_1 \circ \bar{S}_2 \circ \bar{S}_4,\\
\bar{S}_{13} &= \bar{S}_1 \circ \bar{S}_3 \circ \bar{S}_4,\\
\bar{S}_{14} &= \bar{S}_2 \circ \bar{S}_3 \circ \bar{S}_4,\\
\bar{S}_{15} &= \bar{S}_1 \circ \bar{S}_2 \circ \bar{S}_3 \circ \bar{S}_4.
\end{aligned} \tag{26}$$

As we can easily check, choosing arbitrarily four different symmetries in $\bar{S}_i \mid i = \overline{1,15}$, and considering them as independent one of another, the above structure is confirmed. The proposition is proved. □

Theorem 7. *The set $G_{LC} = \{I\} \cup \{\bar{S}_i \mid i = \overline{1,15}\}$ (I being the identity again), endowed with the composition law "$\circ$", form a symmetric, commutative group.*

Proof. Perusing the transformations $\bar{S}_i$, $i = \overline{1,15}$, and taking into account Proposition 9, the following composition table is easy to construct.

$$\begin{bmatrix}
\circ & I & \bar{S}_1 & \bar{S}_2 & \bar{S}_3 & \bar{S}_4 & \bar{S}_5 & \bar{S}_6 & \bar{S}_7 & \bar{S}_8 & \bar{S}_9 & \bar{S}_{10} & \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{S}_{15}\\
I & I & \bar{S}_1 & \bar{S}_2 & \bar{S}_3 & \bar{S}_4 & \bar{S}_5 & \bar{S}_6 & \bar{S}_7 & \bar{S}_8 & \bar{S}_9 & \bar{S}_{10} & \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{S}_{15}\\
\bar{S}_1 & \bar{S}_1 & I & \bar{S}_5 & \bar{S}_6 & \bar{S}_7 & \bar{S}_2 & \bar{S}_3 & \bar{S}_4 & \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_8 & \bar{S}_9 & \bar{S}_{10} & \bar{S}_{15} & \bar{S}_{14}\\
\bar{S}_2 & \bar{S}_2 & \bar{S}_5 & I & \bar{S}_8 & \bar{S}_9 & \bar{S}_1 & \bar{S}_{11} & \bar{S}_{12} & \bar{S}_3 & \bar{S}_4 & \bar{S}_{14} & \bar{S}_6 & \bar{S}_7 & \bar{S}_{15} & \bar{S}_{10} & \bar{S}_{13}\\
\bar{S}_3 & \bar{S}_3 & \bar{S}_6 & \bar{S}_8 & I & \bar{S}_{10} & \bar{S}_{11} & \bar{S}_1 & \bar{S}_{13} & \bar{S}_2 & \bar{S}_{14} & \bar{S}_4 & \bar{S}_5 & \bar{S}_{15} & \bar{S}_7 & \bar{S}_9 & \bar{S}_{12}\\
\bar{S}_4 & \bar{S}_4 & \bar{S}_7 & \bar{S}_9 & \bar{S}_{10} & I & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_1 & \bar{S}_{14} & \bar{S}_2 & \bar{S}_3 & \bar{S}_{15} & \bar{S}_5 & \bar{S}_6 & \bar{S}_8 & \bar{S}_{11}\\
\bar{S}_5 & \bar{S}_5 & \bar{S}_2 & \bar{S}_1 & \bar{S}_{11} & \bar{S}_{12} & I & \bar{S}_8 & \bar{S}_9 & \bar{S}_6 & \bar{S}_7 & \bar{S}_{15} & \bar{S}_3 & \bar{S}_4 & \bar{S}_{14} & \bar{S}_{13} & \bar{S}_{10}\\
\bar{S}_6 & \bar{S}_6 & \bar{S}_3 & \bar{S}_{11} & \bar{S}_1 & \bar{S}_{13} & \bar{S}_8 & I & \bar{S}_{10} & \bar{S}_5 & \bar{S}_{15} & \bar{S}_7 & \bar{S}_2 & \bar{S}_{14} & \bar{S}_4 & \bar{S}_{12} & \bar{S}_9\\
\bar{S}_7 & \bar{S}_7 & \bar{S}_4 & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_1 & \bar{S}_9 & \bar{S}_{10} & I & \bar{S}_{15} & \bar{S}_5 & \bar{S}_6 & \bar{S}_{14} & \bar{S}_2 & \bar{S}_3 & \bar{S}_{11} & \bar{S}_8\\
\bar{S}_8 & \bar{S}_8 & \bar{S}_{11} & \bar{S}_3 & \bar{S}_2 & \bar{S}_{14} & \bar{S}_6 & \bar{S}_5 & \bar{S}_{15} & I & \bar{S}_{10} & \bar{S}_9 & \bar{S}_1 & \bar{S}_{13} & \bar{S}_{12} & \bar{S}_4 & \bar{S}_7\\
\bar{S}_9 & \bar{S}_9 & \bar{S}_{12} & \bar{S}_4 & \bar{S}_{14} & \bar{S}_2 & \bar{S}_7 & \bar{S}_{15} & \bar{S}_5 & \bar{S}_{10} & I & \bar{S}_8 & \bar{S}_{13} & \bar{S}_1 & \bar{S}_{11} & \bar{S}_3 & \bar{S}_6\\
\bar{S}_{10} & \bar{S}_{10} & \bar{S}_{13} & \bar{S}_{14} & \bar{S}_4 & \bar{S}_3 & \bar{S}_{15} & \bar{S}_7 & \bar{S}_6 & \bar{S}_9 & \bar{S}_8 & I & \bar{S}_{12} & \bar{S}_{11} & \bar{S}_1 & \bar{S}_2 & \bar{S}_5\\
\bar{S}_{11} & \bar{S}_{11} & \bar{S}_8 & \bar{S}_6 & \bar{S}_5 & \bar{S}_{15} & \bar{S}_3 & \bar{S}_2 & \bar{S}_{14} & \bar{S}_1 & \bar{S}_{13} & \bar{S}_{12} & I & \bar{S}_{10} & \bar{S}_9 & \bar{S}_7 & \bar{S}_4\\
\bar{S}_{12} & \bar{S}_{12} & \bar{S}_9 & \bar{S}_7 & \bar{S}_{15} & \bar{S}_5 & \bar{S}_4 & \bar{S}_{14} & \bar{S}_2 & \bar{S}_{13} & \bar{S}_1 & \bar{S}_{11} & \bar{S}_{10} & I & \bar{S}_8 & \bar{S}_6 & \bar{S}_3\\
\bar{S}_{13} & \bar{S}_{13} & \bar{S}_{10} & \bar{S}_{15} & \bar{S}_7 & \bar{S}_6 & \bar{S}_{14} & \bar{S}_4 & \bar{S}_3 & \bar{S}_{12} & \bar{S}_{11} & \bar{S}_1 & \bar{S}_9 & \bar{S}_8 & I & \bar{S}_5 & \bar{S}_2\\
\bar{S}_{14} & \bar{S}_{14} & \bar{S}_{15} & \bar{S}_{10} & \bar{S}_9 & \bar{S}_8 & \bar{S}_{13} & \bar{S}_{12} & \bar{S}_{11} & \bar{S}_4 & \bar{S}_3 & \bar{S}_2 & \bar{S}_7 & \bar{S}_6 & \bar{S}_5 & I & \bar{S}_1\\
\bar{S}_{15} & \bar{S}_{15} & \bar{S}_{14} & \bar{S}_{13} & \bar{S}_{12} & \bar{S}_{11} & \bar{S}_{10} & \bar{S}_9 & \bar{S}_8 & \bar{S}_7 & \bar{S}_6 & \bar{S}_5 & \bar{S}_4 & \bar{S}_3 & \bar{S}_2 & \bar{S}_1 & I
\end{bmatrix}$$

This proves the theorem. □

Corollary 5. *The group G_{LC} is endowed with the Boolean structure.*

Proof. As in the cases presented in the previous sections, the composition table points out the fact that every element of G_{LC} is its own inversion with respect to the composition law of the group. □

7. CONCLUDING REMARKS

Examining what we have pointed out in this paper, we can formulate

Theorem 8. *The groups of symmetries G, G^{pol}, G_0, G_∞ are diffeomorphic.*

Proof. The transformations we used to pass from one of the above groups to another being all real analytic diffeomorphisms, the theorem is proved. □

Remark 6. In spite of the above equivalence, G_0 and G_∞ are formally more general. They also cover the limiting situations of collision/ejection and escape/capture, respectively. On the other hand, G and G^{pol} are closer to a physical description of the motion, due to the use of natural (Cartesian or polar) coordinates and of the physical time.

Remark 7. Comparing the symmetries S_i, S_i^{pol} $(i = \overline{1,3})$ with the symmetries $\bar{S}_i$ $(i = \overline{1,3})$, it is clear that they have the same physical significance, respectively.

The symmetries pointed out in this paper are of much help in understanding various properties of the global flow of both the general problem or a concrete problem at hand. Indeed, for each solution proved to exist, they show the existence of many other solutions. Moreover, these symmetries are very useful to find symmetric periodic orbits − especially by means of the continuation method − in perturbed problems depending on a small parameter ε such that for $\varepsilon = 0$ we recover the unperturbed problem. Such questions will be treated elsewhere.

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ЗОНСКИ САТЕЛИТСКИ ПРОБЛЕМ. III. СИМЕТРИЈЕ

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Оригинални научни рад

Дат је осврт на проблем двају тела у вези са пољем силе описаним потенцијалом облика $U = \sum_{k=1}^{n} a_k/r^k$ (r = растојање између честица, a_k = реални параметри) са само једног становишта симетрија. Такве симетрије, изражене у Хамилтоновим координатама, или у стандардним поларним координатама, дају се рекурентно за координате МекГијевог типа обе врсте – судар-експлозија и бесконачност-експлозија. Оне образују дифеоморфне комутативне групе са буловском структуром. Изражен у координатама Леви-Цивита проблем показује велику групу симетрија које су такође комутативне и представљају буловску структуру.