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THE ZONAL SATELLITE PROBLEM. I. NEAR-COLLISION FLOW

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SUMMARY: The force field described by a potential function of the form $U = \sum_{k=1}^{n} a_k/r^k$ (r = distance between particles, a_k = real parameters) models various concrete situations belonging to astronomy, physics, mechanics, astrodynamics, etc. The two-body problem is being tackled in such a field. The motion equations and the first integrals of energy and angular momentum are established. The McGehee-type coordinates are used to blow up the collision singularity and to paste the resulting manifold on the phase space. The flow on the collision manifold is depicted. Then, using the rotational symmetry of the problem and the angular momentum integral, the local flow near collision is described and interpreted in terms of physical motion.

1. INTRODUCTION

This paper starts a research intended to bring a unifying standpoint for several problems of particle nonlinear dynamics. Many classical models, as those of Kepler, Manev, Schwarzschild, Fock, Coulomb, Van der Waals, etc., are included. We deal with two-body problems associated to potentials of the type $U = \sum_{k=1}^{n} a_k/r^k$, in which r is the distance between particles, whereas a_k, $k = \overline{1,n}$, are real constants. Since the best known potential of this kind is that represented by the zonal part of a planetary gravitational potential, we shall assign the name of *zonal satellite problem* to this class of problems (see also Cid *et al.* 1983).

The study of the zonal satellite problem is important especially for physics, mechanics, astronomy, astrodynamics, and not only. Its importance is emphasized by the multitude of concrete situations modellable in this way. It is clear that $a_1 > 0$, $a_k = 0$, ($k = \overline{2,n}$), features the Newtonian field (for a photogravitational field with Newtonian gravitational component, a_1 can equally be negative). The motion in Manev's field ($a_1 > 0$, $a_2 > 0$, $a_k = 0$, $k = \overline{3,n}$) or in Manev-type fields ($a_1, a_2 \in \mathbf{R}$, $a_k = 0$, $k = \overline{3,n}$) also belongs to this model (see, e.g., Diacu 1993, 1996; Diacu *et al.* 1995, Mioc and Stoica 1995a,b, 1996, 1997; Delgado *et al.* 1996; Stoica and Mioc 1997b). The Schwarzschild ($a_1 > 0$, $a_2 = 0$, $a_3 > 0$, $a_k = 0$, $k = \overline{4,n}$) or Schwarzschild-type ($a_1, a_3 \in \mathbf{R}$, $a_2 = 0$, $a_k = 0$, $k = \overline{4,n}$) models constitute further examples (see, e.g., Moeckel 1992; Stoica and Mioc 1997a). Moreover, the zonal satellite problem models the motion in other relativistic fields, as for instance Fock's one (Mioc 1994), or that featured by the Reissner-Nordström metric. The Coulombian field (Sommerfeld 1951; Belenkii 1981) also joins the model. Implications in astrophysics (e.g. Stoica and Mioc 1997a), mechanics (e.g. Moser 1975; McGehee 1981; Diacu 1990), celestial mechanics and dynamical astronomy (Blaga and Mioc 1992; Diacu *et al.*

1995; Mioc and Stoica 1995c; Stoica and Mioc 1996, 1997b), even in atomic physics (Diacu 1993), are to be added to the above arguments.

The present paper attempts to provide a first qualitative insight into the model of the zonal satellite problem. In Section 2 we reduce the problem to a central force problem, write the equation of motion, and point out the first integrals of energy and angular momentum. The qualitative analysis starts with Section 3, in which the powerful tool of McGehee's (1974) technique is used to blow up the collision singularity and to provide new, regularized equations of motion.

In Section 4, consequent on McGehee's transformations, the *collision manifold* (M_0) is pasted, instead of the singularity, on the phase space. It proves to be a 2D torus imbedded in the full 4D phase space if $a_n > 0$, and the empty set (collisionless motion) for $a_n < 0$. The flow on the M_0 torus (consisting of two circles of degenerate equilibria, connected by heteroclinic orbits moving from the upper circle to the lower one) is deprived of physical significance. However, due to the continuity of solutions with respect to initial data, its study provides valuable informations about the local flow near collision.

The behaviour of the solutions in the immediate neighbourhood of collision is tackled in Section 5. Exploiting successively the rotational symmetry of the global flow (characteristic to the zonal satellite problem) and the angular momentum integral, we restrict the full phase space to the 3D reduced phase space, then to dimension 2 (the phase plane - PP). This allows a clear description of the local structure of PP near collision, for the whole possible interplay between the parameters a_{n-1} and a_n. The description is completed by a physical interpretation of the respective phase curves.

2. EQUATIONS OF MOTION AND FIRST INTEGRALS

It is clear that the potential we deal with is central, therefore the associated two-body problem can be reduced to a central force problem. The motion is confined to a plane, so we fix one particle (hereafter *centre*) at the origin of this plane $\mathbf{R}^2$, and study the relative motion of the other particle. Within this framework, denoting by $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2$ the position (or configuration) vector of the particle, the potential reads

$$U(\mathbf{q}) = \sum_{k=1}^{n} a_k / |\mathbf{q}|^k . \tag{1}$$

Let us introduce the momentum vector $\mathbf{p} = \dot{\mathbf{q}}$, $\mathbf{p} = (p_1, p_2) \in \mathbf{R}^2$, and denote by $T(\mathbf{p}) = |\mathbf{p}|^2/2$ the kinetic energy of the particle. In canonical formalism, the equations of motion read

$$\begin{aligned} \dot{\mathbf{q}} &= \partial H(\mathbf{q}, \mathbf{p}) / \partial \mathbf{p}, \\ \dot{\mathbf{p}} &= -\, \partial H(\mathbf{q}, \mathbf{p}) / \partial \mathbf{q}, \end{aligned} \tag{2}$$

defining a vector field on the phase space $\mathbf{Q} \times \mathbf{P}$, where $\mathbf{Q} = \mathbf{R}^2 \setminus \{(0,0)\}$ is the configuration space, whereas $\mathbf{P} = \mathbf{R}^2$ is the momentum space. The Hamiltonian function has the form

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{p}) - U(\mathbf{q}) = |\mathbf{p}|^2/2 - \sum_{k=1}^{n} a_k / |\mathbf{q}|^k . \tag{3}$$

Standard results of the differential equations theory ensure, for given initial conditions $(\mathbf{q}, \mathbf{p})(0) \in \mathbf{Q} \times \mathbf{P}$, the existence and uniqueness of a real analytic solution $(\mathbf{q}, \mathbf{p})$ of equations (2), defined locally on some time interval (t^-, t^+), $-\infty \le t^- < 0 < t^+ \le \infty$. Due to the symmetry, we may study, without loss of generality, the properties of the solution on $[0, t^+)$ only (i.e., the motion in the future). It is clear that $[0, t^+)$ can now be extended to a maximal interval $[0, t^*)$, $t^+ \le t^* \le \infty$.

Using the standard technique, we find that the Hamiltonian has the property

$$H(\mathbf{q}, \mathbf{p}) = h/2 = const., \tag{4}$$

namely is a first integral of the system (called the integral of energy; h is the energy constant). The field being central, the angular momentum is conserved, hence we can obtain another first integral

$$L(\mathbf{q}, \mathbf{p}) = q_1 p_2 - q_2 p_1 = C = const., \tag{5}$$

where C is the constant of angular momentum.

In an explicit form, the equations of motion are

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{p}, \\ \dot{\mathbf{p}} &= -\sum_{k=1}^{n} (k a_k / |\mathbf{q}|^{k+2}) \mathbf{q}, \end{aligned} \tag{6}$$

with the energy relation

$$|\mathbf{p}|^2 - 2 \sum_{k=1}^{n} a_k / |\mathbf{q}|^k = h. \tag{7}$$

3. McGEHEE'S TRANSFORMATIONS

The potential (1) has an isolated singularity for $t = t^* < \infty$, at $\mathbf{q} = (0,0)$. Using a Painlevé-type criterion (e.g. Diacu 1992), it is easy to prove that this singularity corresponds to the collision particle-centre. To remove it and to regularize equations (6), we shall resort to McGehee-type transformations of the second kind (McGehee 1974).

For the first step we use the real analytic diffeomorphism

$$\begin{aligned} \mathbf{Q} \times \mathbf{P} \times [0, t^*) \to &(0, \infty) \times [0, 2\pi] \times \mathbf{R} \times \mathbf{R} \times \\ &\times [0, t^*) \\ ((q_1, q_2), (p_1, p_2); t) \mapsto &(r, \theta, \xi, \eta; t) \end{aligned}$$

defined by

$$\begin{aligned} r &= |\mathbf{q}|, \\ \theta &= \arctan(q_2/q_1), \\ \xi &= \dot{r} = (q_1p_1 + q_2p_2)/|\mathbf{q}| \\ \eta &= r\dot{\theta} = (q_1p_2 - q_2p_1)/|\mathbf{q}|, \end{aligned} \tag{8}$$

which introduces the standard coordinates and the polar components of the velocity. In variables (8), the motion equations acquire the form

$$\begin{aligned} \dot{r} &= \xi, \\ \dot{\theta} &= \eta/r, \\ \dot{\xi} &= \eta^2/r - \sum_{k=1}^{n} k a_k / r^{k+1}, \\ \dot{\eta} &= -\xi\eta/r, \end{aligned} \tag{9}$$

while the first integrals (7) and (5) become respectively

$$\xi^2 + \eta^2 - 2\sum_{k=1}^{n} a_k/r^k = h; \tag{10}$$

$$r\eta = C. \tag{11}$$

The singularity corresponds now to $r = 0$. For the second step, we have to scale down the components of the velocity via the real analytic diffeomorphism

$$\begin{aligned} (0,\infty) \times [0,2\pi] \times \mathbf{R} \times \mathbf{R} \times [0,t^*) \to (0,\infty)\times \\ \times[0,2\pi] \times \mathbf{R}\times\mathbf{R} \times [0,t^*) \\ (r,\theta,\xi,\eta;t) \mapsto (r,\theta,x,y;t) \end{aligned}$$

defined by

$$\begin{aligned} x &= r^{n/2}\xi, \\ y &= r^{n/2}\eta. \end{aligned} \tag{12}$$

The equations of motion (9) now read

$$\begin{aligned} \dot{r} &= x/r^{n/2}, \\ \dot{\theta} &= y/r^{n/2+1}, \\ \dot{x} &= (nx^2/2 + y^2)/r^{n/2+1} - \sum_{k=1}^{n} k a_k / r^{k+1-n/2}, \\ \dot{y} &= (n/2-1)xy/r^{n/2+1}, \end{aligned} \tag{13}$$

whereas the first integrals (10) and (11) become respectively

$$x^2 + y^2 = hr^n + 2\sum_{k=1}^{n} a_k r^{n-k}; \tag{14}$$

$$y^2 = C^2 r^{n-2}. \tag{15}$$

The singularity at $r = 0$ still persists. To remove it, we introduce the timelike variable s through the real analytic diffeomorphism

$$\begin{aligned} (0,\infty) \times [0,2\pi] \times \mathbf{R} \times \mathbf{R} \times [0,t^*) \to [0,\infty)\times \\ \times[0,2\pi] \times \mathbf{R}\times\mathbf{R} \times [0,\infty) \\ (r,\theta,x,y;t) \mapsto (r,\theta,x,y;s) \end{aligned}$$

defined by

$$ds = r^{-n/2-1}dt. \tag{16}$$

With (16), and keeping by abuse the same notations for the new functions of the fictitious time s, the motion equations (13) become

$$\begin{aligned} r' &= rx, \\ \theta' &= y, \\ x' &= nx^2/2 + y^2 - \sum_{k=1}^{n} k a_k r^{n-k}, \\ y' &= (n/2-1)xy, \end{aligned} \tag{17}$$

where $' = d/ds$. Obviously, the integrals (14) and (15) keep their expressions.

4. COLLISION MANIFOLD

Both the equations of motion (17) and the first integrals (14)-(15) are now well defined for the boundary $r = 0$. This means that the phase space can be analytically extended to contain the manifold

$$[M_{\mathrm{col}} = \{(r,\theta,x,y) \mid r = 0\},]$$

which is invariant under the flow because $r' = 0$ for $r = 0$. The relations (14) and (15) also extend smoothly to this boundary.

Having in view (14), let us define the constant energy manifold

$$[M_h = \{(r,\theta,x,y) \mid x^2 + y^2 = hr^n + 2\sum_{k=1}^{n} a_k r^{n-k}\},]$$

which corresponds to a fixed level of energy. Now we are able to define the *collision manifold*

$$\begin{aligned} M_0 &= M_{\mathrm{col}} \cap M_h = \\ &= \{(r,\theta,x,y) \mid r = 0, \theta \in S^1, x^2 + y^2 = 2a_n\}. \end{aligned} \tag{18}$$

Since the last term in the expansion of the potential (1) must obviously be nonzero, the case $a_n = 0$ will not be considered. For $a_n < 0$, M_0 is the empty set; the particle cannot encounter collisions. Analogous results were established by Saari (1974), or Stoica and Mioc (1997a) within different contexts.

Consider hence $a_n > 0$; by (18), M_0 is a 2D cylinder in the 3D space of the coordinates $(\theta, x, y) \in S^1 \times \mathbf{R} \times \mathbf{R}$ (Figure 1). But, since $\theta \in S^1$ (the segment $[0, 2\pi]$ with the end points pasted together), the M_0 cylinder may be identified with a 2D torus, both actually being imbedded in the 4D full phase space of the McGehee- type coordinates (r, θ, x, y).

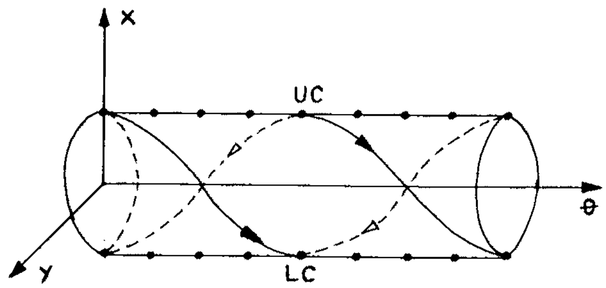

Fig. 1. *The M_0 cylinder and the flow on it for* $n = 4$.

So, by means of McGehee's technique, we have blown up the singularity and pasted the M_0 torus, instead of it, on the phase space. Having in view (16), M_0 is formed only by equilibria of the flow on the full phase space. In other words, every collisional phase trajectory needs an infinite amount of fictitious time s to reach M_0.

In the sequel we shall describe the flow on M_0. This flow, although deprived of physical significance, provides valuable informations about the behaviour of near-collision orbits (due to the continuity of solutions with respect to the initial conditions).

Using (17) and (14), the vector field on M_0 reads

$$\begin{aligned} \theta' &= y, \\ x' &= -(n/2-1)y^2, \\ y' &= (n/2-1)xy. \end{aligned} \tag{19}$$

One sees immediately that equations (19) admit the fixed points $(\theta, x, y) = (\theta_e, \pm\sqrt{2a_n}, 0)$, with arbitrary $\theta \in S^1$. Explicitly, there are two circles of degenerate equilibria on the M_0 torus: the *upper circle* (UC): $(\theta \in S^1, x = \sqrt{2a_n}, y = 0)$, and the *lower circle* (LC): $(\theta \in S^1, x = -\sqrt{2a_n}, y = 0)$. By (19), removing the cases $n = 1$ (Newton-type field) and $n = 2$ (Manev-type field), one sees that $x' < 0$ for $y \neq 0$, which means that all orbits of the flow on M_0 are heteroclinic and move from UC to LC. To determine the shape of these trajectories, let us introduce the variable α via $x = \sqrt{2a_n}\cos\alpha$, $y = \sqrt{2a_n}\sin\alpha$. Taking into account (19), this leads to $d\alpha/d\theta = n/2 - 1$.To exemplify, the phase curves on M_0 for $n = 4$ are illustrated in Figure 1.

This provides a complete qualitative image of the flow on M_0.

To add some issues within this framework, focus on formula (15). On the one hand, it is clear that all collisional trajectories eject from UC or tend to LC. On the other hand, collisions occur not only for $C = 0$, hence for radial motion (as in the Newtonian case), but also for $C \neq 0$. This is the so-called *black hole effect* (the particle spirals infinitely many times around the centre before collision/after ejection) pointed out by Wintner (1941), McGehee (1981), Diacu *et al.* (1995), or Stoica and Mioc (1997a).

As a final remark, in case M_0 is nonempty, it does not depend on h, hence every total energy level shares this boundary.

5. NEAR-COLLISION FLOW

In this section we shall study the local flow in the neighbourhood of the collision manifold. First observe that θ does not appear explicitly in either the regularized vector field (17) or first integrals (14)-(15). We can hence reduce the dimension of the phase space from 4 to 3, by factorizing the flow to S^1. Of course, every phase curve in this 3D reduced phase space (hereafter RPS) actually is a manifold consisting of the product between the respective orbit and S^1. The M_0 torus in full phase space reduces in RPS to the circle $\tilde{M}_0 = \{r = 0, x^2 + y^2 = 2a_n\}$.

We can further reduce RPS to dimension 2 by resorting to the integral of angular momentum (15), which relates y to r. So we obtain the phase plane of the coordinates (r, x) (hereafter PP). It is obvious that what we said above about the orbits in RPS is to be applied to the solutions in PP, too. The $\tilde{M}_0$ circle in RPS reduces in PP to the points M $(r = 0, x = \sqrt{2a_n})$ and N $(r = 0, x = -\sqrt{2a_n})$.

Under these conditions, the energy integral in PP reads

$$x^2 = \sum_{j=o}^{n} A_j r^{n-j}, \tag{20}$$

where $A_0 = h$, $A_1 = 2a_1$, $A_2 = 2a_2 - C^2$, $A_j = 2a_j$ $(j = \overline{3, n})$. It is obvious that we consider $a_n \neq 0$ and very small r (neglecting the terms in r^i, $i \geq 2$).

The local phase portrait for $a_n > 0$ is given in Figure 2a. For $a_{n-1} < 0$ we have the curves 1: heteroclinic trajectories which eject from collision, reach a maximum distance $r_{\max} = -a_n/a_{n-1}$, then tend back to collision. For $a_{n-1} = 0$, we have the curves 2 and 2' (which eject from collision or tend to collision, respectively). For $a_{n-1} > 0$, the phase portrait consists of the curves 3 and 3', which have similar significances as the curves 2 and 2', respectively. Of course, the ejection/collision is asymptotic. The phase portrait also contains the points M and N: degenerate orbits which remain forever in collision (in this first approximation with local character).

In case $a_n < 0$, the motion is collision-free. If $a_{n-1} < 0$ the real motion is impossible in our approximation. If $a_{n-1} > 0$, the phase portrait in PP is given in Figure 2b: the phase trajectories approach the centre, reach a minimum distance $r_{\min} = -a_n/a_{n-1}$, then move away from the centre.

The near-collision physical orbits behave like the above described PP trajectories. It is easy to see that the motion is radial for $C = 0$, and spiral for $C \neq 0$ (with black hole effect at collision/ejection in the case of collisional motion).

These results provide a qualitative picture of the phase and physical motion in the neighbourhood of collision within the framework of the zonal satellite problem.

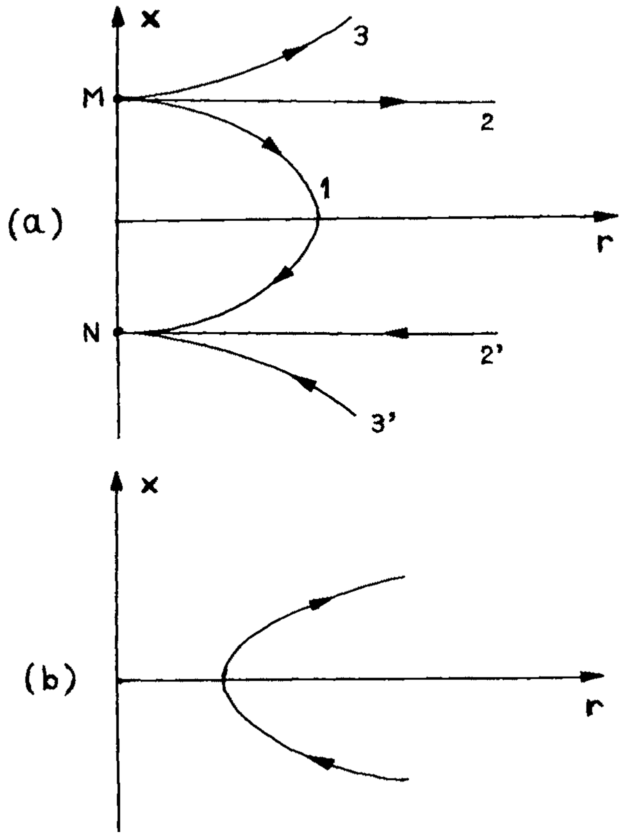

Fig. 2. *The near-collision flow in PP for: (a)* $a_n > 0$ *(b)* $a_n < 0$.

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ПРОБЛЕМ ЗОНСКИХ САТЕЛИТА. I. ПРОТОК У ОБЛАСТИ БЛИЗУ СУДАРА

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Поље силе описано потенцијалом облика $U = \sum_{k=1}^{n} a_k/r^k$ (r - растојање између материјалних тачака, a_k = реални параметри) моделује разне конкретне ситуације које спадају у астрономију, физику, механику, астродинамику итд. У таквом пољу се разматра проблем двају тела. Установљени су једначине кретања и први интеграли енергије и момента импулса. Коришћене су координате МекГеeовог типа у циљу одстрањивања сударне сингуларности и пресликавања резултујућег скупа на фазни простор. Описано је протицање на сударном скупу. Тада, коришћењем обртне симетрије проблема и интеграла момента импулса описује се и интерпретира локални проток у области близу судара преко физичког кретања.